PHYS:7729 Homework #8

Due at the beginning of class, Thursday, April 27, 2023.

1. (10 pts) Recall the Child-Langmuir Law for a 1-D electrostatic plasma of hydrogen with isothermal electrons with temperature T_e and cold ions,

$$j_i = \frac{4}{9}\epsilon_0 \left(\frac{2e}{m_i}\right)^{1/2} \frac{\phi_w^{3/2}}{d^2}, \tag{1}$$

which expresses the space-charge limited ion current across the sheath in the limit $-e\phi_w/T_e \gg 1$ as a function of sheath width d and the potential difference ϕ_w between the wall and the potential at the sheath edge $x = d$. Recall that the potential at the sheath edge is chosen to define a potential of zero, $\phi(d) = 0$. Note that we absorb Boltzmann's constant to give temperature in units of energy.

   (a) Taking the ion current to be given by $j_i = en_d c_s$, where $c_s = \sqrt{T_e/m_i}$ and the n_d is ion density at the sheath edge, compute an expression for the sheath width d as a function of the wall potential ϕ_w, the electron temperature T_e, and the Debye length computed using the plasma conditions at the sheath edge $x = d$.

   (b) For typical laboratory plasma parameters of $T_e = 5\ eV$ and $n_d = 10^{18}$ m^3, compute the width of the sheath for a wall voltage of $\phi_w = -300$ V.

2. (10 pts) For a Langmuir probe trace using a cylindrical probe (for which the electron saturation current does not become constant), (a) compute the electron temperature (in eV) using the data in the table below, (b) estimate the plasma potential ϕ_p, and (c) estimate floating potential ϕ_f.

Probe Bias (V)	- Probe Current (A)
-65.00	-0.0001290
-60.00	-0.0001290
-55.00	-0.0000860
-50.00	-0.0000860
-45.00	-0.0000430
-40.00	-0.0000430
-35.00	0.0000000
-30.00	0.0000430
-26.00	0.0001730
-24.00	0.0003020
-22.00	0.0004740
-20.00	0.0009060
-18.00	0.0015960
-16.00	0.0032350
-14.00	0.0041410
-12.00	0.0046580
-10.00	0.0051330
-8.00	0.0055210
-6.00	0.0058230
-4.00	0.0062540
-2.00	0.0064270
0.00	0.0068150

3. (10 pts) Dimensional Analysis for Turbulence with Dissipation
Consider the case of incompressible turbulence (so that the mass density ρ_0 of the fluid remains constant). Let us apply the Buckingham Pi Theorem to this problem to determine the minimum number of dimensionless parameters upon which the system depends.

Consider the following physical quantities of the system: (i) fluid velocity U, (ii) length scale of the fluid flow L, (iii) turnaround time τ for an eddy of length scale L and fluid velocity U, (iv) incompressible energy cascade rate ϵ (which has units of velocity squared over time), (v) kinematic viscosity ν, (vi) equilibrium pressure of the fluid p_0, and (vii) equilibrium density ρ_0. Note that the sound speed c_s in fluid is given by $c_s^2 = \gamma p_o/\rho_0$, were γ is the adiabatic index.

(a) Determine the number of physical quantities n and the number of primary quantities k upon which those physical quantities depend. Upon how many dimensionless quantities π_j, where $j = 1, \ldots, m$, does the system depend? In other words, what are the values of n, k, and m?

(b) Write down the m dimensionless quantities π_j upon which the system depends.
HINT: You may want to look at Lecture #21 Sections III.A and III.B for guidance on the most appropriate way to group the physical quantities into the dimensionless π_j.

(c) For each of the dimensionless π_j, identify which property of the turbulent system is associated with that π_j.

4. (10 pts) Nondimensionalization of the Linearized Shallow Water Equations
The Shallow Water Equations are a two-dimensional system of equations describing the dynamics of motions with wavelengths greater than the depth of the water H in a rotating frame of reference, valid for wavenumbers $kH \ll 1$. They are useful in describing large-scale ocean dynamics. The linearized equations are

$$\frac{\partial \eta}{\partial t} + H\nabla \cdot \mathbf{U}_1 = 0$$

$$\frac{\partial \mathbf{U}_1}{\partial t} + f_0 \hat{\mathbf{z}} \times \mathbf{U}_1 + g\nabla\eta = 0$$

Here η is the fluctuation of the sea surface due to waves from mean sea level and g is the gravitational acceleration. The term $f_0 \hat{\mathbf{z}} \times \mathbf{U}_1$ represents the Coriolis force due to the rotation of the Earth, where the frequency f_0 represents the effectively constant frequency of this rotation at a given latitude.

(a) We want to determine the scaling of the characteristic wave phase velocity c and characteristic wave number k on the physical quantities of the system. Upon which physical quantities of the system are c and k expected to depend?

(b) According to the Buckingham Pi Theorem, how many dimensionless parameters m are expected to govern the wave dynamics of the system?

(c) Normalize all length scales by the inverse wavenumber $1/k$, the velocity by the phase velocity of the waves c, and the time by $1/(kc)$. Thus, for example, the dimensionless velocity is given by $\hat{\mathbf{U}}_1 = c\mathbf{U}_1$. Convert the equations into dimensionless form, boxing the final result for each equation.

(d) Identify the m dimensionless parameters that remain in the dimensionless equations.
HINT: The remaining coefficients in the equations after nondimensionalization are these dimensionless parameters.