

# Lecture #13 Landau Damping of Electrostatic Waves

Howes ①

## I. Laplace-Fourier Solution of Electrostatic Plasma Waves

### A. Setup:

1. Electrostatic:  $\underline{E} = -\nabla\phi$ ,  $\underline{B} = 0$ ,  $\underline{E}_0 = 0 \Rightarrow \phi_0 = 0$

2. Vlasov-Maxwell System:

$$\frac{\partial f_s}{\partial t} + \underline{v} \cdot \nabla f_s - \frac{q_s}{m_s} \nabla\phi \cdot \frac{\partial f_s}{\partial \underline{v}} = 0$$

$$-\nabla^2 \phi = \frac{1}{\epsilon_0} \sum_s \int d^3v q_s f_s$$

3. Take  $\underline{k} = k \hat{z}$

B. Linearization 1.  $f_s = f_{s0}(\underline{v}) + \epsilon f_{s1}(\underline{x}, \underline{v}, t)$   
 $\phi = \phi_0 + \epsilon \phi_1(\underline{x}, t)$

2. At  $\mathcal{O}(\epsilon)$ : a.  $\frac{\partial f_{s1}}{\partial t} + \underline{v} \cdot \nabla f_{s1} - \frac{q_s}{m_s} \nabla\phi_1 \cdot \frac{\partial f_{s0}}{\partial \underline{v}} = 0$

b.  $-\nabla^2 \phi_1 = \frac{1}{\epsilon_0} \sum_s \int d^3v q_s f_{s1}$

### C. Fourier Transform in Space Only $\nabla \Rightarrow i\mathbf{k}$

1. a.  $\frac{\partial f_{s1}}{\partial t} + i \underline{v} \cdot \mathbf{k} f_{s1} - i \frac{q_s \phi_1}{m_s} \mathbf{k} \cdot \frac{\partial f_{s0}}{\partial \underline{v}} = 0$

b.  $k^2 \phi_1 = \frac{1}{\epsilon_0} \sum_s \int d^3v q_s f_{s1}$

### D. Laplace Transform in Time: $\tilde{f}_s(p) = \int_0^\infty dt f_s(t) e^{-pt}$

1. a.  $\tilde{f}'_s(p) + i \underline{v} \cdot \mathbf{k} \tilde{f}_s(p) - i \frac{q_s \tilde{\phi}_1(p)}{m_s} \mathbf{k} \cdot \frac{\partial f_{s0}}{\partial \underline{v}} = 0$

b. Using  $\tilde{f}'(p) = p\tilde{f}(p) - f(0)$ , we get

$$(p + i \underline{v} \cdot \mathbf{k}) \tilde{f}_s(p) = i \frac{q_s \tilde{\phi}_1(p)}{m_s} \mathbf{k} \cdot \frac{\partial f_{s0}}{\partial \underline{v}} + f(0)$$

# Lecture #13 (Continued)

Howes (2)

## I.O.D. (Continued)

### 2. Solving for $\tilde{f}_s(p)$

$$\tilde{f}_s(p) = \frac{i k \cdot \frac{\partial f_{s0}}{\partial \underline{v}} \frac{q_s \tilde{\phi}_1(p)}{m_s}}{p + i k \cdot \underline{v}} + \frac{f_s(0)}{p + i k \cdot \underline{v}}$$

The poles in this solution are due to  $\tilde{\phi}_1(p)$  poles and  $p = -i k \cdot \underline{v}$

### E. Substitute $\tilde{f}_s(p)$ into Poisson's Equation to Solve for $\tilde{\phi}_1(p)$

$$1. \quad k^2 \tilde{\phi}_1 = \frac{1}{\epsilon_0} \sum_s \int d^3 \underline{v} q_s \left\{ \frac{i k \cdot \frac{\partial f_{s0}}{\partial \underline{v}} \frac{q_s \tilde{\phi}_1(p)}{m_s}}{p + i k \cdot \underline{v}} + \frac{f_s(0)}{p + i k \cdot \underline{v}} \right\}$$

NOTE:  $\tilde{\phi}_1(p)$  does not depend on  $\underline{v}$ .

2. Divide by  $k^2$  and collect  $\tilde{\phi}_1(p)$  terms:

$$a. \quad \tilde{\phi}_1 \left[ 1 - \sum_s \frac{(q_s^2 n_0)}{\epsilon_0 m_s} \int d^3 \underline{v} \frac{i k \cdot \frac{\partial f_{s0}}{\partial \underline{v}}}{p + i k \cdot \underline{v}} \right] = \frac{1}{k^2 \epsilon_0} \sum_s \int d^3 \underline{v} \frac{q_s f_s(0)}{p + i k \cdot \underline{v}}$$

Dispersion Relation  $D(p, \underline{k})$ 
Initial Conditions  $N(p, \underline{k})$

b. Solution to  $D(p, \underline{k}) = 0$  gives normal modes of the system.

$$c. \quad \text{Thus } \tilde{\phi}_1(p) = \frac{N(p, \underline{k})}{D(p, \underline{k})}$$

d. Inverse Laplace Transform  $\tilde{\phi}_1(p)$  by Residue Theorem is due to poles in  $N(p, \underline{k})$  and zeros of  $D(p, \underline{k})$

### F. Simplify Using $\underline{k} = k \hat{\underline{z}}$ and Reduced Distribution Function $f_{s0}(v_z)$

$$1. \quad f_{s0}(v_z) \equiv \frac{1}{n_0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^2 \underline{v}_\perp f_{s0}(\underline{v})$$

Lecture #13 (Continued)  
 I. F. (Continued)

Howes (3)

2. Thus a.  $D(p, k) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{i k \frac{\partial f_{s0}}{\partial v_z}}{p + i k v_z} = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\frac{\partial f_{s0}}{\partial v_z}}{v_z - \frac{i p}{k}}$

b. Similarly

$N(p, k) = \sum_s \frac{-i q_s n_0}{\epsilon_0 k^3} \int_{-\infty}^{\infty} dv_z \frac{F_s(v)}{v_z - \frac{i p}{k}}$

3. Our solution  $\tilde{\phi}(k, p)$  is then given by

Pieces of Solution due to:

$$\tilde{\phi}(p, k) = \frac{-i \sum_s \frac{q_s n_0}{\epsilon_0 k^3} \int_{-\infty}^{\infty} dv_z \frac{F_s(v)}{v_z - \frac{i p}{k}}}{1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\frac{\partial f_{s0}}{\partial v_z}}{v_z - \frac{i p}{k}}}$$

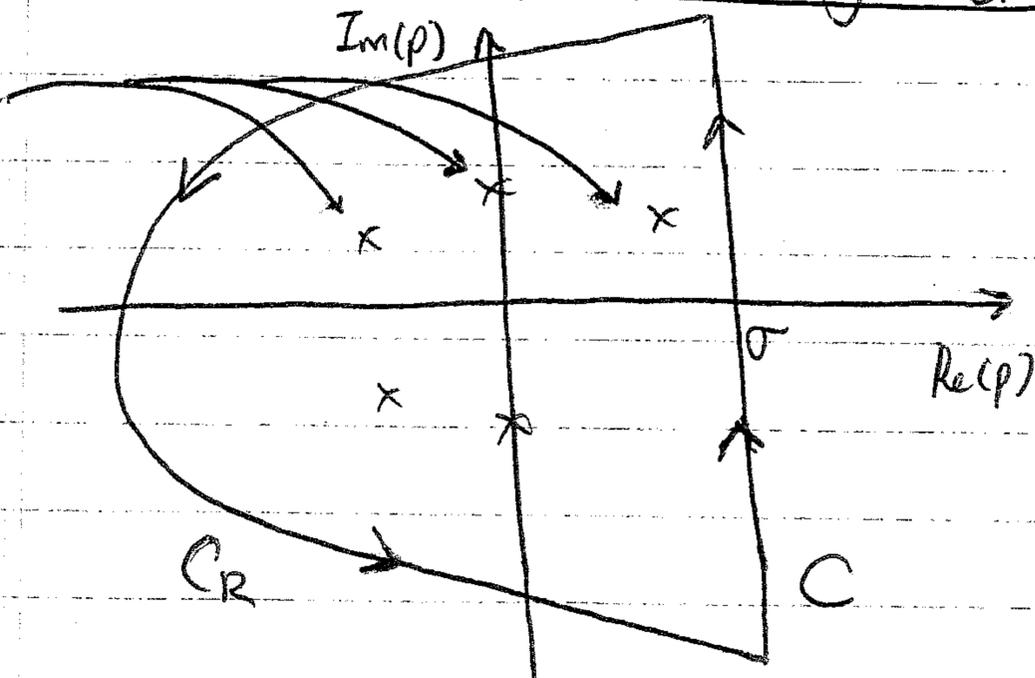
$\left. \begin{array}{l} \text{Poles in Numerator} \\ \text{zeros in Denominator} \\ D(p, k) = 0 \text{ is} \\ \text{Normal Modes!} \end{array} \right\}$

4. We want to find  $\phi(k, t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} dp \tilde{\phi}(k, p) e^{pt}$

Using the Residue Theorem.

G. Evaluation of  $\phi(k, t)$  Using Residue Theorem

Poles of  $\tilde{\phi}(p, k)$



1. To Evaluate  $\phi(k, t)$  using the Residue Theorem, we close the contour by completing the loop at  $\text{Re}(p) \rightarrow -\infty$  (This is section C<sub>R</sub>)

Thus  $\int_C dp \tilde{\phi}(k, p) e^{pt} = \int_{\sigma-i\infty}^{\sigma+i\infty} dp \tilde{\phi}(k, p) e^{pt} + \int_{C_R} dp \tilde{\phi}(k, p) e^{pt}$

## I. G. (Continued)

2. a. To evaluate contour integral using the Residue Theorem requires that  $\tilde{\Phi}(k, p)$  be analytic within and on contour  $C$ .

b. But, the function  $\tilde{\Phi}(k, p)$  was only defined for  $\text{Re}(p) > 0$ .

$\Rightarrow$  Thus we must analytically continue  $\tilde{\Phi}(k, p)$  to the negative Real half plane  $\text{Re}(p) < 0$ .

c. This is not straight forward due to the  $\sqrt{z}$ -integral in both  $D(p, k)$  and  $N(p, k)$ . For example,

$$D(p, k) = 1 - \sum_S \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dz \frac{\partial F_{s0} / \partial z}{\sqrt{z} - \frac{ip}{k}}$$

d. This function is discontinuous on the line  $\text{Re}(p) = 0$ .

Why? ① Remember  $p = \gamma - i\omega$ , so the denominator is

$$\sqrt{z} - \frac{1}{k}(\gamma - i\omega) = \sqrt{z} - \frac{\omega}{k} - \frac{i\gamma}{k}$$

② If  $\text{Re}(p) = \gamma = 0$ , then we have  $\int_{-\infty}^{\infty} dz \frac{\partial F_{s0} / \partial z}{\sqrt{z} - \frac{\omega}{k}}$

and the integral becomes undefined at  $\sqrt{z} = \frac{\omega}{k}$ .

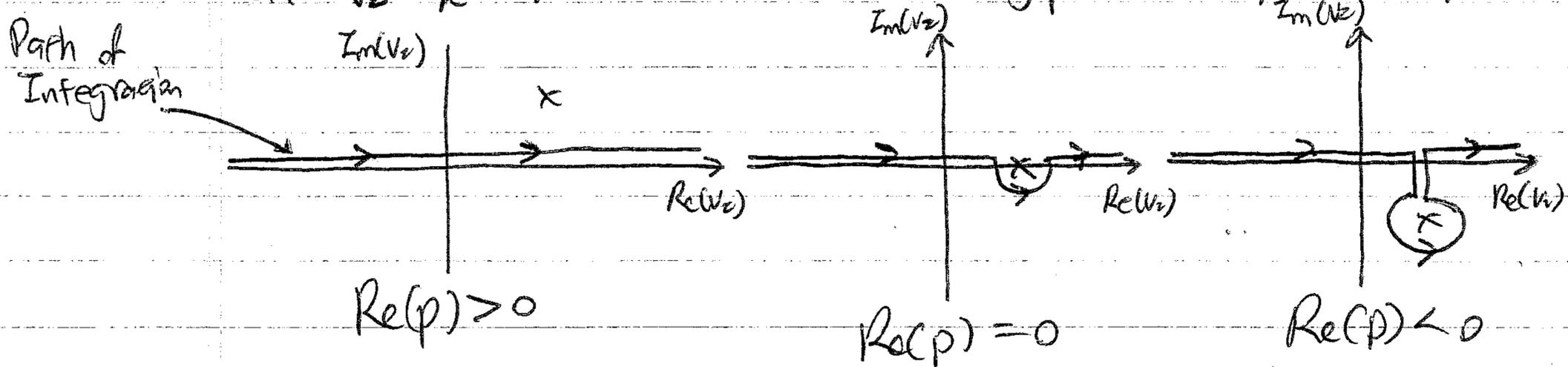
e. Since we must perform our contour integral over the entire complex plane  $p$ , this problem at  $\text{Re}(p) = 0$  must be resolved.

### H. Landau's Analytic Continuation of $D(p, k)$ and $N(p, k)$

1. Landau solved this problem by carrying out a careful analytic continuation of  $D(p, k)$  and  $N(p, k)$  to  $\text{Re}(p) < 0$ .

I. H. (Continued)

2. Consider the case  $k > 0$  ( $k < 0$  is analogous). The pole at  $v_z = \frac{i p}{k}$  then lies at the following points in complex  $v_z$  space.



a. Treating the integral  $\int_{-\infty}^{\infty} dv_z$  as a contour integration in complex  $v_z$  space, Landau discovered the contour of integration so that it always passes below the pole in  $v_z$  space

b. In this way, the functions  $D(p, k)$  and  $N(p, k)$  [and thus  $\tilde{\phi}(p, k)$ ] are analytically continued into the  $\text{Re}(p) < 0$  half of the complex  $p$  plane.

c. Now we can go ahead and use the Residue Theorem to evaluate  $\int_{\sigma-i\infty}^{\sigma+i\infty} \tilde{\phi}(p, k) e^{p t}$ .

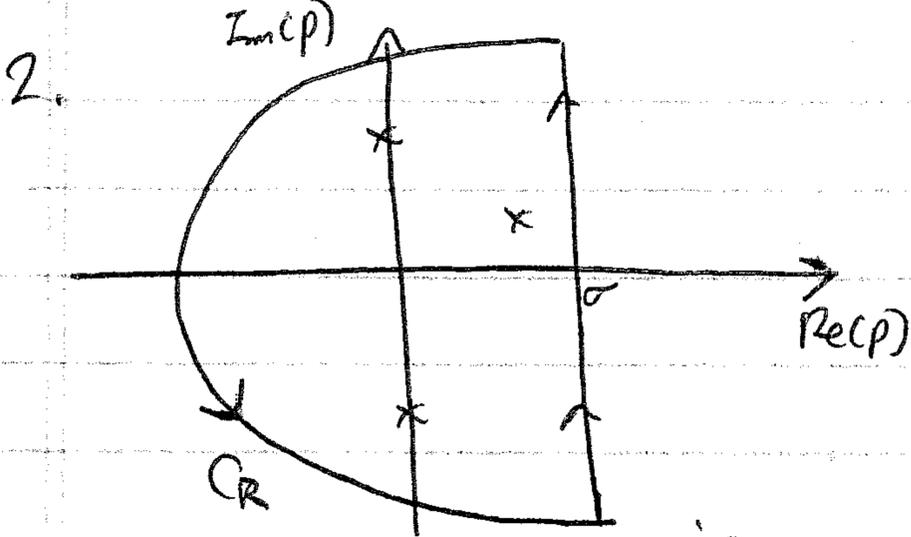
3. a. We'll look at concrete examples of this  $v_z$  integration soon.

b. For Maxwellian equilibrium distribution, this gives rise to the Plasma Dispersion Function.

I. Evaluation of  $\phi(k, t)$

i. Remember 
$$F(t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \tilde{\phi}(p) e^{p t}$$

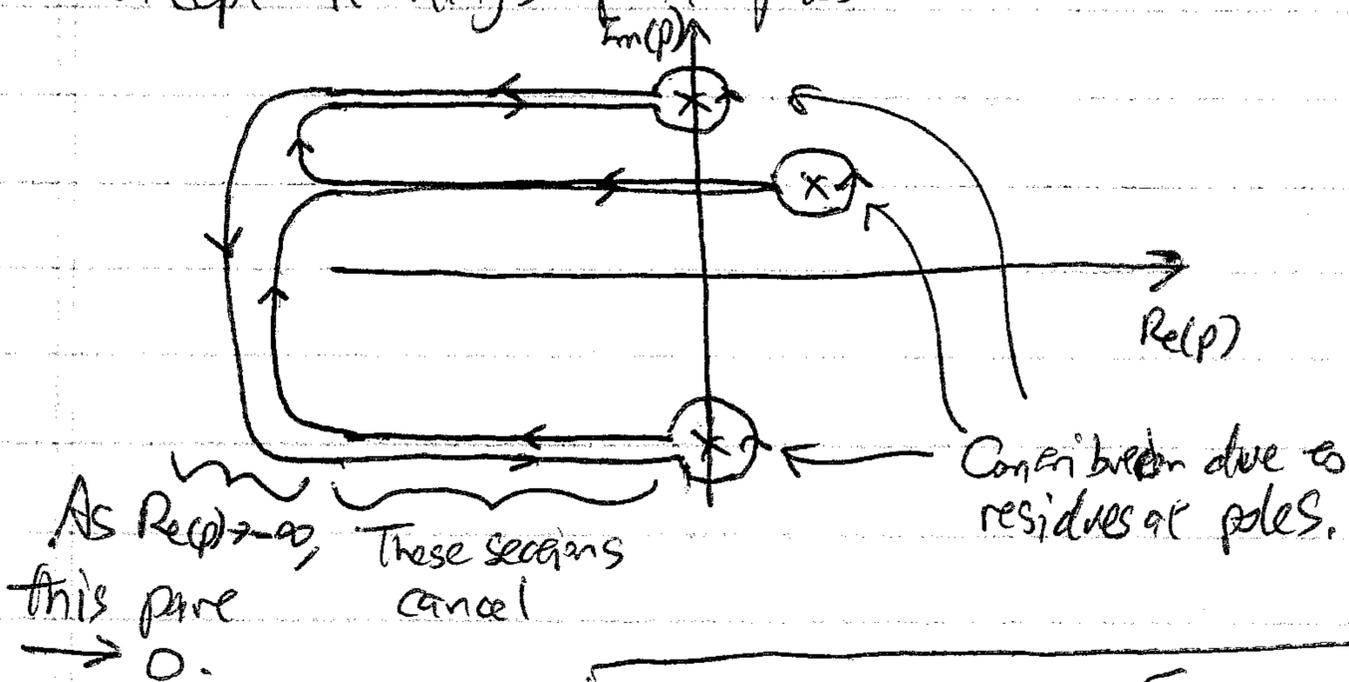
I. I. (Continued)



a.

$$\int_C dp \tilde{\phi}(k, p) e^{pt} = \underbrace{\int_{\sigma-i\infty}^{\sigma+i\infty} \tilde{\phi}(k, p) e^{pt}}_{= 2\pi i \phi(k, t)} + \underbrace{\int_{C_R} dp \tilde{\phi}(k, p) e^{pt}}_{\text{As } \text{Re}(p) \rightarrow -\infty} = 2\pi i \sum_{\substack{p=p_j \\ \text{Res}[\tilde{\phi}(k, p) e^{pt}]}}$$

b. By deformation of paths, we can see that the only contribution to integral is due to residues. Deform  $C$  to  $\text{Re}(p) \rightarrow -\infty$ , except if hangs up at poles:



3. Thus, we find

$$\phi(k, t) = \sum_j \text{Res} \left[ \tilde{\phi}(k, p) e^{pt} \right]$$

4. Remember,  $p$ 's are complex,  $p = \gamma - i\omega$ , so solutions typically have a behavior,  $\sim e^{\gamma t} e^{-i\omega t}$ , oscillatory with frequency  $\omega$  and a growth rate for  $\gamma > 0$ , or damping rate for  $\gamma < 0$ .

II. Solution for Cauchy Velocity Distribution

A. Cauchy Velocity Distribution

1. A simple analytical distribution function is

DEF: Cauchy Reduced Velocity Distribution  $F_0(v_z) = \frac{C}{\pi} \left( \frac{1}{C^2 + v_z^2} \right)$

a. NOTE:  $\int_{-\infty}^{\infty} dv_z F_0(v_z) = 1$

2. Consider ions immobile, so  $F_{0i} = F_{0e}$  and  $n_i = 0$ .

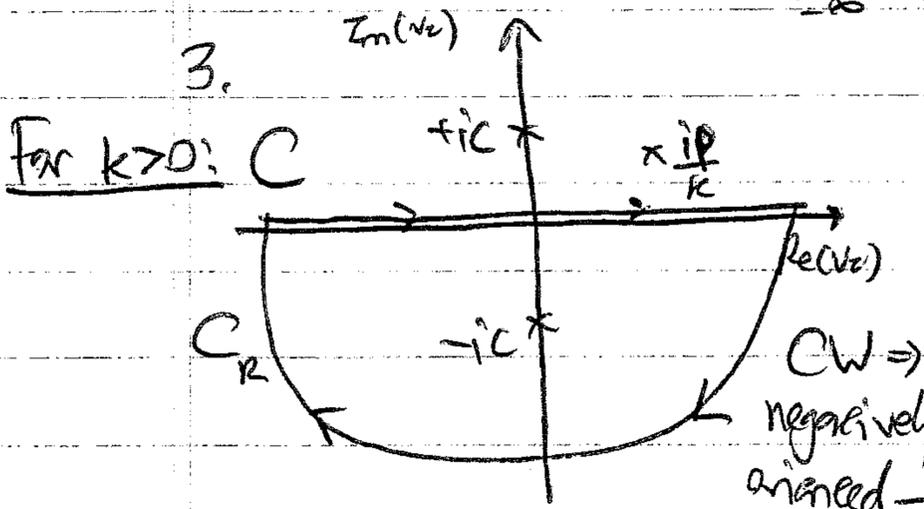
B. Velocity Integral over  $v_z$

1. Our Dispersion Relation is  $D(p, k) = 1 - \frac{\omega_p^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\partial F_0 / \partial v_z}{v_z - \frac{i p}{k}}$

where we only consider the electron contribution since ions are immobile.

2. We can integrate by parts (as done in lect #11, II. F. 3.) to yield

$$D(p, k) = 1 - \frac{\omega_p^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{F_0}{(v_z - \frac{i p}{k})^2} = 1 - \frac{\omega_p^2 C}{k^2 \pi} \int_{-\infty}^{\infty} dv_z \frac{1}{(v_z - iC)(v_z + iC)(v_z - \frac{i p}{k})^2}$$



a. Close at  $Im(v_z) \rightarrow -\infty$

b. Let  $g(v_z) = \frac{1}{(v_z - iC)(v_z + iC)(v_z - \frac{i p}{k})^2}$

c. Thus  $\int_C dv_z g(v_z) = \int_{-\infty}^{\infty} dv_z g(v_z) + \int_{CR} dv_z g(v_z)$

as  $Im(v_z) \rightarrow -\infty$   
(Really  $|v_z| \rightarrow \infty$ )

$$= -2\pi i \sum_j \text{Res}[g(v_z)]_{v_z = v_{zj}}$$

d. Thus for pole at  $v_z = -iC$

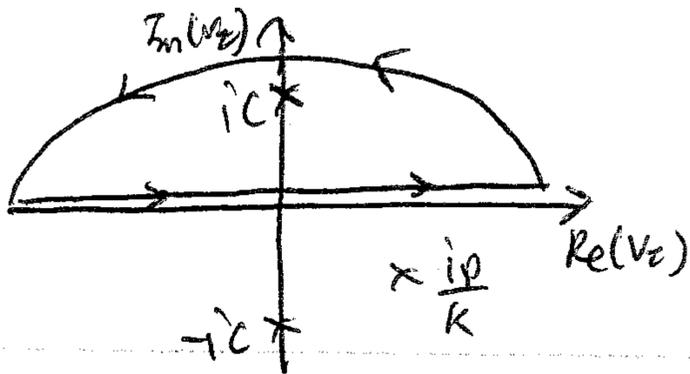
$$= -2\pi i \frac{1}{(-2iC)(-iC - \frac{i p}{k})^2} = \frac{\pi}{C} \frac{-1}{(C + \frac{p}{k})^2}$$

e. So, we find for  $k > 0$ :

$$D(p, k) = 1 + \frac{\omega_p^2 C}{k^2 \pi} \frac{\pi}{C} \frac{+1}{(C + \frac{p}{k})^2} = 1 + \frac{\omega_p^2}{(p + kC)^2}$$

Lecture # 13 (Continued)

II. B. (Continued).



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4. Similarly for  $k < 0$

a. Close in upper half plane  $\text{Im}(v_z) \rightarrow \infty$  (CCW orientation).

b. Thus 
$$\int_{-\infty}^{\infty} dv_z g(v_z) = 2\pi i \sum_j \text{Res}[g(v_z)] \rightarrow \text{pole at } v_z = +ic$$

$$= 2\pi i \frac{1}{2ic \left( ic - \frac{ip}{k} \right)^2} = \frac{-\pi}{c \left( c - \frac{p}{k} \right)^2}$$

c. Thus  $D(p, k) = 1 + \frac{\omega_p^2}{(p - kc)^2}$

5. Noting that for  $k > 0$ ,  $k = |k|$  and for  $k < 0$ ,  $k = -|k|$ , we can write these as a single equation

$$D(p, k) = 1 + \frac{\omega_p^2}{(p + |k|c)^2} = 0$$

b. NOTE: Since this solution is a polynomial, analytic continuation to the  $\text{Re}(p) < 0$  plane is trivial.

6. Roots of dispersion relation are  $p = -|k|c \pm i\omega_p$

C. Solving for  $N(k, p)$

Initial condition on  $f_s$

1. 
$$N(k, p) = -i \sum_s \frac{q_s n_{0s}}{\epsilon_0 k^3} \int_{-\infty}^{\infty} dv_z \frac{F_s(k, v, 0)}{v - ip/k}$$

a. If we have a specific form for the initial conditions  $F_s(k, v, 0)$ , then we can perform the integral analogous to the procedure above.

b. An important point is that, as long as  $F_s(k, v, 0)$  do not have any singularities or discontinuities, the result of the integration will not have any singularities.  $\Rightarrow$  Thus, no poles in  $N(k, p)$

## II. C. (Continued)

2. Rather than solve for a specific form of  $F_S(\underline{k}, \underline{v}, 0)$ , we note

$$\tilde{\phi}(\underline{k}, p) \underbrace{D(\underline{k}, p)}_{\substack{\text{Dispersion} \\ \text{Relation}}} = \underbrace{N(\underline{k}, p)}_{\substack{\text{Initial} \\ \text{Conditions}}} \quad (\text{see I.E. 2, a. earlier})$$

a. We simply denote  $N(\underline{k}, p) = \frac{1}{\omega_p} \phi(\underline{k}, 0)$  since it is determined by the initial conditions.

b. Thus  $\tilde{\phi}(\underline{k}, p) = \frac{\phi(\underline{k}, 0)}{\omega_p D(p, \underline{k})} = \frac{\phi(\underline{k}, 0)}{\omega_p \left(1 + \frac{\omega_p^2}{(p + k/c)^2}\right)} = \frac{(p + k/c)^2 \phi(\underline{k}, 0)}{[(p + k/c)^2 + \omega_p^2] \omega_p}$

D. Completing Solution for  $\phi(\underline{k}, t)$ 

1. As we solved earlier (I. I. 3.),  $\phi(\underline{k}, t) = \sum_j \text{Res}_{p=p_j} \left[ \tilde{\phi}(\underline{k}, p) e^{pt} \right]$

a. Here  $\tilde{\phi}(\underline{k}, p) e^{pt} = \frac{(p + k/c)^2 \phi(\underline{k}, 0) e^{pt}}{(p + k/c - i\omega_p)(p + k/c + i\omega_p) \omega_p}$

Poles are roots  $p = -k/c + i\omega_p$  &  $p = -k/c - i\omega_p$

2. Thus 
$$\begin{aligned} \phi(\underline{k}, t) &= \frac{(-k/c + i\omega_p + k/c)^2 \phi(\underline{k}, 0) e^{-k/c t} e^{i\omega_p t}}{(-k/c + i\omega_p + k/c + i\omega_p) \omega_p} \\ &+ \frac{(-k/c - i\omega_p + k/c)^2 \phi(\underline{k}, 0) e^{-k/c t} e^{-i\omega_p t}}{(-k/c - i\omega_p + k/c - i\omega_p) \omega_p} \\ &= \frac{-\omega_p^2 \phi(\underline{k}, 0) e^{-k/c t} e^{i\omega_p t}}{2i\omega_p^2} + \frac{-\omega_p^2 \phi(\underline{k}, 0) e^{-k/c t} e^{-i\omega_p t}}{-2i\omega_p^2} \end{aligned}$$

$$\phi(\underline{k}, t) = -\phi(\underline{k}, 0) e^{-k/c t} \left( \frac{e^{i\omega_p t} - e^{-i\omega_p t}}{2i} \right) = -\phi(\underline{k}, 0) \sin(\omega_p t) e^{-k/c t}$$