29:293 Homework #9

Due at the beginning of class, Thursday, April 30, 2015.

1. Hydrodynamic Keplerian Accretion Disk
   Calculate the dispersion relation for a hydrodynamic disk in Keplerian rotation about a central body of mass M. Assume incompressible motion $\nabla \cdot \mathbf{U} = 0$ and a wave vector $\mathbf{k} = k\hat{\mathbf{z}}$ that varies only in the z direction (aligned with the axis of the Keplerian rotation). Take the accretion disk to be an isothermal, thin disk.

   (a) Write down the relevant first-order hydrodynamic equations (having removed the equilibrium) based on the assumptions above.

   (b) Why do the pressure gradient and gravitational force terms in the momentum equation not contribute to the first order equations?

   (c) Show that the dispersion relation for this system is

   $$\omega^2 = 4\Omega^2 + \frac{d\Omega^2}{d\ln R}.$$

   (d) Use the definition of the epicyclic frequency κ to show that this dispersion relation may be alternatively written as $\omega^2 = \kappa^2$.

   (e) Show that this implies a stability criterion $dL/dR > 0$ for stability and that the Keplerian disk is stable. Here $L = R^2\Omega$ is the specific angular momentum.

2. Growth Rates of the Magnetorotational Instability
   In a magnetized Keplerian accretion disk, the dispersion relation for fluctuations with $\mathbf{k} = k\hat{\mathbf{z}}$ in the incompressible limit is

   $$\omega^4 - \omega^2(\kappa^2 + 2k^2 v_A^2) + k^2 v_A^2 \left(k^2 v_A^2 + \frac{d\Omega^2}{d\ln R} \right) = 0.$$

   (a) Determine the maximum unstable growth rate $\gamma_{max} = \mathrm{Im}(\omega)$ for an arbitrary unstable rotation profile $\Omega(R)$ with $d\Omega/dR < 0$.

   (b) Calculate the wavenumber $(kv_A)^2_{max}$ at which this maximum growth rate occurs.

   (c) For a Keplerian rotation profile $\Omega^2 = GM/R^3$, calculate the values of γ_{max} and $(kv_A)_{max}$ in terms of the angular rotation frequency Ω.