PHYS:4731 Homework #2

Due at the beginning of class, Friday, September 13, 2024.

1. Here we will apply a simplified version of Multiple-Timescale Analysis to the problem of particle motion in constant, uniform $\mathbf{E}$ and $\mathbf{B}$ fields.

   As done in Lecture #3, we assume a right-handed, orthonormal basis aligned with the direction of the magnetic field $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{b}})$ such that $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{b}}$. The Lorentz Force Law is

   $$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

   for an electric field $\mathbf{E} = E_1\hat{\mathbf{e}}_1 + E_2\hat{\mathbf{e}}_2 + E_{\|}\hat{\mathbf{b}}$ and a magnetic field $\mathbf{B} = B_0\hat{\mathbf{b}}$. For this problem, we will take the case $E_{\|} = 0$.

   (a) First, let us convert the dimensional form of the Lorentz Force Law above to a dimensionless equation. Derive the dimensionless form
   $$\frac{d\mathbf{v}'}{dt'} = \mathbf{E}' + \mathbf{v}' \times \hat{\mathbf{b}} \tag{1}$$
   for dimensionless quantities $t' = \omega_c t$, $\mathbf{v}' = \mathbf{v}/v_\perp$, and $\mathbf{E}' = \frac{\mathbf{E}}{B_0 v_\perp}$ where $v_\perp = \sqrt{v_1^2 + v_2^2}$.

   (b) Verify that the quantity $E' = |\mathbf{E}'|$ is dimensionless (in the SI system of units).

   (c) Show that the condition $E' \ll 1$ means that the $\mathbf{E} \times \mathbf{B}$ drift is slow compared to the perpendicular velocity, $|\mathbf{v}_E| \ll v_\perp$.

   (d) Assuming $E' \ll 1$, the timescales of the Larmor motion and the $\mathbf{E} \times \mathbf{B}$ drift are well separated. For the expansion parameter, take $\epsilon = E' \ll 1$. As an aid in the bookkeeping for the order of magnitude of each term, we can add an ϵ to the electric field term in our equation to remind us of its order,
   $$\frac{d\mathbf{v}'}{dt'} = \epsilon\mathbf{E}' + \mathbf{v}' \times \hat{\mathbf{b}} \tag{2}$$
   We'll assume a fast timescale t' and a slow timescale $\tau' = \epsilon t'$. Decompose the total velocity into rapidly varying piece $\mathbf{v}_1'$ and a smaller slowly varying piece $\mathbf{v}_2'$, $\mathbf{v}' = \mathbf{v}_1'(t') + \epsilon\mathbf{v}_2'(\tau')$.
   Write down the expansion of d/dt' assuming two timescales.

   (e) Derive the equation at $\mathcal{O}(1)$ and solve for $\mathbf{v}_1'(t')$ given the (dimensional) initial conditions at $t = 0$ of $\mathbf{v} = v_\perp\hat{\mathbf{e}}_1 + v_{\|0}\hat{\mathbf{b}}$.

   (f) Derive the equation at $\mathcal{O}(\epsilon)$. Solve for $\mathbf{v}_2'(\tau')$. HINT: Do not forget to treat t' and τ' as independent variables.

   (g) Sum the solution for each order to get the total solution $\mathbf{v}'(t', \tau')$. Convert back to dimensional form to yield the final, complete solution $\mathbf{v}(t)$.

2. A cylindrical column of plasma rotates around its central axis (as though it were a rigid solid) at an angular velocity ω_0. A constant uniform magnetic field $\mathbf{B}_0$ is present parallel to the axis of rotation.

   (a) Assuming that the rigid rotation can be described by $\mathbf{v}_E = (\omega_0\hat{\mathbf{z}}) \times \mathbf{r}$, where $\mathbf{v}_E = \mathbf{E} \times \mathbf{B}/B^2$, compute the electric field $\mathbf{E}$ in the plasma column. Use cylindrical (r, ϕ, z) coordinates.

   (b) Is there a polarization charge $\rho = \epsilon_0\nabla \cdot \mathbf{E}$ associated with this electric field? If so, how does ρ depend on the distance from the central axis?

   (c) Find the electrostatic potential Φ such that $\mathbf{E} = -\nabla\Phi$.

   (d) How would you induce a motion of this type in a magnetized plasma column?