# Lecture #9: Particle Motion in Slowly Varying $\underset{\sim}{E}$-Fields 
Polarization Drift

## I. Polarization Drift

A. Consider an Electric Field varying slowly in time $\underset{\sim}{E}(\tau)$ with a constant Magnetic Field $\underset{\sim}{B} = B_0 \hat{\underset{\sim}{b}}$.

1. NOTE: $\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{j} + \epsilon_0 \mu_0 \frac{\partial \underset{\sim}{E}}{\partial t}$, we assume $|\underset{\sim}{j}| \gg |\epsilon_0 \frac{\partial \underset{\sim}{E}}{\partial t}|$, so $\frac{dB}{dt} \simeq 0$, thus $\underset{\sim}{B} \simeq$ constant!

B. Multiple Time Scale Analysis

1. $$\frac{d\underset{\sim}{v}}{dt} = \frac{q}{m}(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B})$$

2. a. Take $\underset{\sim}{E}(\tau)$ varies only on slow timescale $\tau = \epsilon t$

b. $\underset{\sim}{v} = \underset{\sim}{v}_1(t) + \epsilon \underset{\sim}{v}_2(\tau) + \epsilon^2 \underset{\sim}{v}_3(\tau) + \ldots$

c. Also assume $\underset{\sim}{E}(\tau) \cdot \underset{\sim}{B} = 0$

3. As before,
$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial \tau}{\partial t}\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \tau}$$

4. We'll take a small electric field such that the $E \times B$ drift velocity $v_E \ll v$, where $v$ is Larmor orbit velocity.

Thus
$$\frac{d\underset{\sim}{v}}{dt} = \frac{q}{m}(\epsilon \underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B})$$

5. Substitute expanded solution:

$$\frac{\partial}{\partial t}(\underset{\sim}{v}_1 + \cancelto{0}{\epsilon \underset{\sim}{v}_2} + \cancelto{0}{\epsilon^2 \underset{\sim}{v}_3}) + \epsilon \frac{\partial}{\partial \tau}(\cancelto{0}{\underset{\sim}{v}_1} + \epsilon \underset{\sim}{v}_2 + \epsilon^2 \underset{\sim}{v}_3) = \epsilon \frac{q}{m}\underset{\sim}{E}(\tau)$$
$$+ \frac{q}{m}(\underset{\sim}{v}_1 + \epsilon \underset{\sim}{v}_2 + \epsilon^2 \underset{\sim}{v}_3) \times \underset{\sim}{B}$$

a. Taking $\underset{\sim}{B} = B_0 \hat{\underset{\sim}{b}}$,

$$\frac{\partial \underset{\sim}{v}_1}{\partial t} + \epsilon^2 \frac{\partial \underset{\sim}{v}_2}{\partial \tau} + \epsilon^3 \frac{\partial \underset{\sim}{v}_3}{\partial \tau} = \epsilon \frac{q \underset{\sim}{E}(\tau)}{m} + \omega_c \underset{\sim}{v}_1 \times \hat{\underset{\sim}{b}} + \epsilon \omega_c \underset{\sim}{v}_2 \times \hat{\underset{\sim}{b}} + \epsilon^2 \omega_c \underset{\sim}{v}_3 \times \hat{\underset{\sim}{b}}$$

I.B. (Continued)

6. $O(1)$: $\frac{\partial \underset{\sim}{v_1}}{\partial t} = \omega_c \underset{\sim}{v_1} \times \underset{\sim}{\hat{b}}$

a. This is just the usual, fast timescale Larmor gyration about the magnetic field.

b. The general solution for this motion can be written

$$\boxed{\underset{\sim}{V_1} = V_\perp \cos(\omega_c t + \phi)\underset{\sim}{\hat{e}_1} - V_\perp \sin(\omega_c t + \phi)\underset{\sim}{\hat{e}_2} + V_{\parallel}\underset{\sim}{\hat{b}}}$$

for a right-handed coordinate system s.t. $\underset{\sim}{\hat{e}_1} \times \underset{\sim}{\hat{e}_2} = \underset{\sim}{\hat{b}}$

7. $O(\epsilon)$: $0 = \cancel{\epsilon}\frac{q}{m}\underset{\sim}{E}(\tau) + \cancel{\epsilon}\frac{qB_0}{m}\underset{\sim}{V_2} \times \underset{\sim}{\hat{b}}$

a. This is just the slow timescale $\underset{\sim}{E} \times \underset{\sim}{B}$ drift.

b. Operating $\underset{\sim}{\hat{b}} \times$ on equation gives:

$$\underset{\sim}{\hat{b}} \times \underset{\sim}{E}(\tau) = \frac{qB_0}{m}\underset{\sim}{\hat{b}} \times (\underset{\sim}{V_2} \times \underset{\sim}{\hat{b}}) = B_0\left(\underset{\sim}{V_2}(\underset{\sim}{\hat{b}} \cdot \underset{\sim}{\hat{b}}) - V_{2\parallel}\underset{\sim}{\hat{b}}\right)$$

or

$$\boxed{\underset{\sim}{V_2} = V_{2\parallel}\underset{\sim}{\hat{b}} + \frac{\underset{\sim}{E}(\tau) \times \underset{\sim}{B}}{B_0^2}}$$

8. $O(\epsilon^2)$: $\cancel{\epsilon^2}\frac{\partial \underset{\sim}{V_2}}{\partial \tau} = \cancel{\epsilon^2}\omega_c \underset{\sim}{V_3} \times \underset{\sim}{\hat{b}}$

a. At this order, the solution $\underset{\sim}{V_2}$ is considered to be known.

Thus, $\frac{\partial \underset{\sim}{V_2}}{\partial \tau} = \cancel{\frac{\partial V_{2\parallel}}{\partial \tau}}^{\,0}\underset{\sim}{\hat{b}} + \frac{1}{B_0^2}\frac{\partial \underset{\sim}{E}}{\partial \tau} \times \underset{\sim}{B} = \frac{\frac{\partial \underset{\sim}{E}}{\partial \tau} \times \underset{\sim}{\hat{b}}}{B_0}$

b. $\frac{1}{B_0}\frac{\partial \underset{\sim}{E}}{\partial \tau} \times \underset{\sim}{\hat{b}} = \omega_c \underset{\sim}{V_3} \times \underset{\sim}{\hat{b}}$

c. Take $\underset{\sim}{\hat{b}} \times$ this equation

$$\frac{1}{B_0}\underset{\sim}{\hat{b}} \times \left(\frac{\partial \underset{\sim}{E}}{\partial \tau} \times \underset{\sim}{\hat{b}}\right) = \frac{1}{B_0}\left[\frac{\partial \underset{\sim}{E}}{\partial \tau}(\underset{\sim}{\hat{b}} \cdot \underset{\sim}{\hat{b}}) - \underset{\sim}{\hat{b}}\cancel{\left(\underset{\sim}{\hat{b}} \cdot \frac{\partial \underset{\sim}{E}}{\partial \tau}\right)}^{\,0}\right] = \frac{1}{B_0}\frac{\partial \underset{\sim}{E}}{\partial \tau}$$

$$\omega_c \underset{\sim}{\hat{b}} \times (\underset{\sim}{V_3} \times \underset{\sim}{\hat{b}}) = \omega_c\left[\underset{\sim}{V_3}(\underset{\sim}{\hat{b}} \cdot \underset{\sim}{\hat{b}}) - \underset{\sim}{\hat{b}}(\underset{\sim}{V_3} \cdot \underset{\sim}{\hat{b}})\right] = \omega_c\left(\underset{\sim}{V_3} - V_{3\parallel}\underset{\sim}{\hat{b}}\right)$$

d. Thus

$$\boxed{\underset{\sim}{V_3} = V_{3\parallel}\underset{\sim}{\hat{b}} + \frac{1}{\omega_c B_0}\frac{\partial \underset{\sim}{E}(\tau)}{\partial \tau}}$$

Lecture #9 (Continued) 

I. B. (Continued)

9. Putting the full solution together: (Taking $v_{\parallel} = v_{2\parallel} = v_{3\parallel} = 0$)

$$\boxed{\underset{\sim}{v} = \underbrace{v_\perp\left(\cos(\omega_c t+\phi)\hat{\underset{\sim}{e}}_1 - \sin(\omega_c t+\phi)\hat{\underset{\sim}{e}}_2\right)}_{\text{Zeroth order Larmor Motion}} + \underbrace{\frac{\underset{\sim}{E}(t)\times\underset{\sim}{B}}{B_0^2}}_{\text{First-order } \underset{\sim}{E}\times\underset{\sim}{B} \text{ drift}} + \underbrace{\frac{1}{\omega_c B_0}\frac{d\underset{\sim}{E}}{dt}}_{\text{Second-order Polarization Drift}}}$$

C. Polarization Drift:

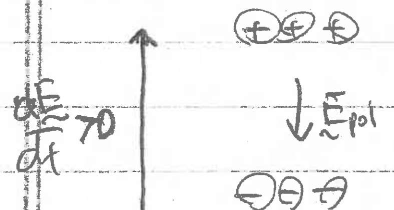
1. For slowly varying electric field $\underset{\sim}{E}(t)$ (slow with respect to the Larmor motion), we define the

Polarization Drift $\boxed{\underset{\sim}{v}_p \equiv \frac{1}{\omega_c B}\frac{d\underset{\sim}{E}}{dt}}$

2. Using $\omega_c = \frac{qB}{m}$, we have

$$\underset{\sim}{v}_p = \frac{m}{qB^2}\frac{d\underset{\sim}{E}}{dt}$$

a. Polarization drift is charge dependent
$\Rightarrow$ ions and electrons drift in opposite directions




b. Resulting polarization of plasma opposes increasing applied Electric Field.

c. Because $m_i \gg m_e$, ions dominate the polarization drift.

3. Polarization Current: $\underset{\sim}{j}_p = \sum_s q_s n_s \underset{\sim}{v}_p = \sum_s \frac{\cancel{q_s} n_s m_s}{\cancel{q_s} B^2}\frac{d\underset{\sim}{E}}{dt}$

a. $\underset{\sim}{j}_p = \sum_s \frac{n_s m_s}{B^2}\frac{d\underset{\sim}{E}}{dt}$

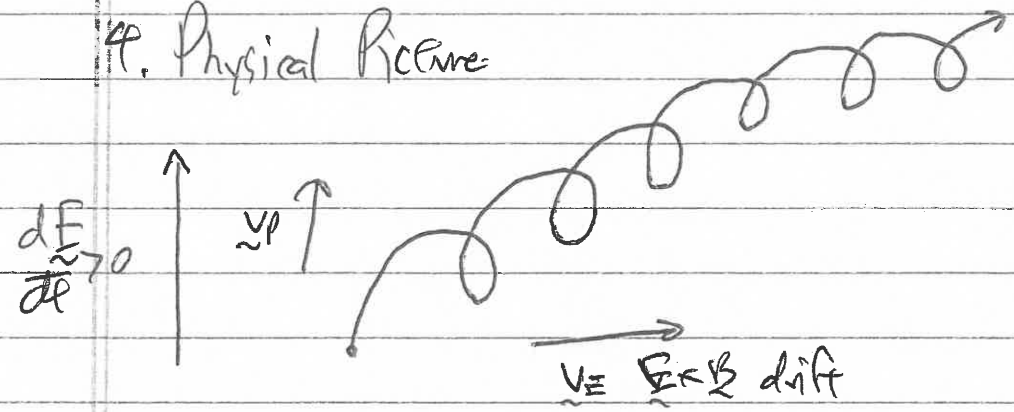
b. Mass dependence means ion contribute most to polarization current.

I.C.3. (Continued)

b. NOTE: $\underset{\sim}{E}\times\underset{\sim}{B}$ velocity is the same for both species, so it cancels, producing no net current.

$$j_E = \sum_s q_s n_s \underset{\sim}{v}_E = \sum_s q_s n_s \frac{\underset{\sim}{E}\times\underset{\sim}{B}}{B^2} = \frac{\underset{\sim}{E}\times\underset{\sim}{B}}{B^2}\underbrace{\cancel{\sum_s q_s n_s}}_{\to 0 \text{ by quasineutrality.}} = 0$$

4. Physical Picture:




5. The Polarization Drift can lead to an increase in energy.

a. $$\frac{d\mathcal{E}}{dt} = \underset{\sim}{v}\cdot\underset{\sim}{F} = \underset{\sim}{v}\cdot q(\underset{\sim}{E} + \cancel{\underset{\sim}{v}\times\underset{\sim}{B}}) = q\underset{\sim}{v}\cdot\underset{\sim}{E}$$

b. $$= q\Big[\cancel{v_\perp\cos(\omega_c t+\phi)\underset{\sim}{E}\cdot\hat{\underset{\sim}{e}}_1 + v_\perp\sin(\omega_c t+\phi)\underset{\sim}{E}\cdot\hat{\underset{\sim}{e}}_2}$$

Average over Larmor orbit $\Rightarrow 0$.

$$+ \cancel{\frac{\underset{\sim}{E}(t)\times\underset{\sim}{B}}{B^2}\cdot\underset{\sim}{E}}^{\,0} + \frac{1}{\omega_c B}\frac{d\underset{\sim}{E}}{dt}\cdot\underset{\sim}{E}\Big]$$

c. Thus $$\frac{d\mathcal{E}}{dt} = \frac{q}{\omega_c B}\frac{d}{dt}\left(\frac{E^2}{2}\right) = \frac{d}{dt}\left[\frac{1}{2}m\left(\frac{E^2}{B^2}\right)\right]$$

d. Note: $|v_E|^2 = \frac{E^2}{B^2}$, so this can be written $\boxed{\frac{d}{dt}\left(\frac{1}{2}m v_E^2\right) = \frac{d\mathcal{E}}{dt}}$

e. **The Polarization Drift leads to the acceleration of particles to achieve the $\underset{\sim}{E}\times\underset{\sim}{B}$ drift velocity.**