

Lecture #1: Waves in a Cold, Uniform Magnetized Plasma

I. The Plasma Conductivity & Dielectric Tensors

A. Cold Plasma Equations:

1. Continuity Eq:  $\frac{\partial n_s}{\partial t} + \underline{v}_s \cdot \nabla n_s = -n_s \nabla \cdot \underline{v}_s$

2. Momentum Eq:  $m_s n_s \left[ \frac{\partial \underline{v}_s}{\partial t} + \underline{v}_s \cdot \nabla \underline{v}_s \right] = q_s n_s (\underline{E} + \underline{v}_s \times \underline{B})$

3. Maxwell's Eqs: Ampere-Maxwell  $\nabla \times \underline{B} = \mu_0 \underline{j} + \mu_0 \epsilon_0 \frac{\partial \underline{E}}{\partial t}$

Faraday

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

Gauss

$$\nabla \cdot \underline{E} = \frac{\rho_q}{\epsilon_0}$$

$$\nabla \cdot \underline{B} = 0$$

$$\underline{j} = \sum_s n_s q_s \underline{v}_s$$

$$\rho_q = \sum_s n_s q_s$$

B. Microscopic vs. Macroscopic Form of Maxwell's Equations

1. The Macroscopic Form of Maxwell's Equations is:

$$\nabla \times \underline{H} = \underline{j}_r + \frac{\partial \underline{D}}{\partial t}$$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\nabla \cdot \underline{D} = \rho_r$$

$$\nabla \cdot \underline{B} = 0$$

where  $\rho_q = \rho_r + \rho_p$  where  $\rho_r \equiv$  "real" charge and  $\rho_p =$  polarization charge

and  $\underline{j} = \underline{j}_r + \underline{j}_m$  where  $\underline{j}_r =$  "real" current and  $\underline{j}_m =$  magnetization current.

2. We can choose all of the plasma charges to be part of  $\rho_p$ .

a. Thus  $\underline{D} = \epsilon_0 \underline{E} + \underline{P}$  where  $\underline{P} \equiv$  Induced dipole moment, or polarization.  
Displacement Field

3. We want to define  $\underline{D}$  in terms of  $\underline{E}$  using the plasma properties. They are related by the Dielectric Tensor,  $\underline{\epsilon}$ .

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L. B. (Continued)

4. Consider the "Macroscopic" version of Ampere/Maxwell Law

$$\nabla \times \underline{H} = \underline{j}_r + \frac{\partial \underline{D}}{\partial t}$$

a. If we consider the plasma response as part of  $\underline{D}$ , then

$$\underline{j}_r = 0,$$

b. The magnetic moment of individual particles in a plasma is typically negligible, so  $\underline{B} = \mu_0 \underline{H}$

c. Thus, we find  $\nabla \times \underline{B} = \mu_0 \frac{\partial \underline{D}}{\partial t} \xrightarrow{\text{Fourier transform}} i \underline{k} \times \underline{B} = -i \omega \mu_0 \underline{D}$

Now, we want to relate this  $\underline{D}$  to plasma electric field  $\underline{E}$ .

C. The Plasma Conductivity Tensor & Plasma Dielectric Tensor

1. The plasma current is given by  $\underline{j} = \sum_s n_s q_s \underline{v}_s$

2. Using the momentum eqs for ions and electrons, we can relate  $\underline{v}_s$  to the Electric field  $\underline{E}$  to yield,

$$\underline{j} = \underline{\sigma} \cdot \underline{E} \quad \leftarrow \text{Gives the response of the plasma to an applied electric field } \underline{E}$$

DEF: Conductivity Tensor:  $\underline{\sigma}$

For linear motions, this is easily determined using momentum equations  
For ions & electrons

3. Now, the microscopic form of Ampere-Maxwell Law is.

$$\nabla \times \underline{B} = \mu_0 \underline{j} + \mu_0 \epsilon_0 \frac{\partial \underline{E}}{\partial t}$$

$$\text{Fourier transform} \Rightarrow i \underline{k} \times \underline{B} = \mu_0 \underline{j} + \mu_0 \epsilon_0 (-i \omega) \underline{E} = \epsilon_0 \mu_0 (-i \omega) \left[ \frac{\underline{j}}{-i \omega \epsilon_0} + \underline{E} \right]$$

4. But, from above, "macroscopic" form gives  $i \underline{k} \times \underline{B} = -i \omega \mu_0 \underline{D}$

5. Thus 
$$\underline{D} = \epsilon_0 \left[ \frac{j\sigma}{-\omega\epsilon_0} + \underline{I} \right] \cdot \underline{E} = \epsilon_0 \left[ \frac{j\sigma}{\omega\epsilon_0} + \underline{I} \right] \cdot \underline{E}$$

DEF: 
$$\underline{D} = \epsilon_0 \underline{\underline{\epsilon}} \cdot \underline{E}$$

where 
$$\underline{\underline{\epsilon}} = \underline{I} + \frac{j\sigma}{\omega\epsilon_0}$$
 is the Dielectric Tensor (Gurnett & Bhattacharjee use  $\underline{K}$ )

D. Homogeneous Wave Equation in terms of Dielectric Tensor

1. Faraday's Law: 
$$\underline{k} \times \underline{E} = \omega \underline{B}$$

Ampere-Maxwell Law: 
$$\underline{k} \times \underline{B} = -\frac{\omega}{c^2} \underline{\underline{\epsilon}} \cdot \underline{E}$$
 (where we have used  $\mu_0 \epsilon_0 = \frac{1}{c^2}$ )

2. Substitute in for  $\underline{B}$  using Faraday's Law:

$$\underline{n} \times (\underline{n} \times \underline{E}) + \underline{\underline{\epsilon}} \cdot \underline{E} = 0$$

DEF: Index of Refraction:

$$\underline{n} \equiv \frac{c \underline{k}}{\omega}$$

3. This equation can be written as a dispersion relation for the electric field in tensor form:

Rank 2 Tensor  
(3x3 matrix)

$$\underline{\underline{D}} \cdot \underline{E} = 0$$

Do not confuse with displacement

field vector,  $\underline{D}$

where  $\underline{\underline{D}} = \underline{\underline{D}}(\omega, \underline{k})$  ← We'll see the matrix form of this soon.

a. NOTE:

We can write  $\underline{n} \times (\underline{n} \times \underline{E}) = n^2 (\underline{\hat{n}} \underline{\hat{n}} - \underline{I}) \cdot \underline{E}$  where  $\underline{\hat{n}} = \frac{\underline{n}}{|\underline{n}|}$

b. The condition for the existence of a non-zero solution to  $\underline{E}$  is

$$\text{Det}(\underline{\underline{D}}) = 0. \text{ (as usual),}$$

Magnetize

## II. The Plasma Conductivity & Dielectric Tensors for a Cold Plasma

### A. The Plasma Conductivity Tensor $\underline{\underline{\sigma}}$

1. We want to calculate  $\underline{\underline{\sigma}}$  and then  $\underline{\underline{\epsilon}}$  for a cold magnetized plasma

2. Let us consider a single species plasma with ions & electrons

Such that  $\sum_s n_{s0} q_s = n_i q_i + n_e q_e = 0$

↑ charge neutrality of equilibrium.

3. We'll use the momentum equation to find the conductivity  $\underline{\underline{\sigma}}$

$$m_s n_s \left[ \frac{\partial \underline{U}_s}{\partial t} + \underline{U}_s \cdot \nabla \underline{U}_s \right] = q_s n_s (\underline{E} + \underline{U}_s \times \underline{B})$$

4. Linearize: a.  $n_s = n_{s0} + n_{s1}$

$$\underline{U}_s = \underline{E} U_{s1} \quad (\text{no zero order } \underline{U}_s)$$

$$\underline{E} = \underline{E}_1 \quad (\text{no zero order } \underline{E})$$

$$\underline{B} = \underline{B}_0 + \underline{E} \underline{B}_1$$

b. We'll take  $\underline{B}_0 = B_0 \hat{z}$

c. Thus, we get

$$\begin{aligned}
 & \epsilon m_s n_{s0} \frac{\partial \underline{U}_{s1}}{\partial t} + \epsilon^2 m_s n_{s1} \frac{\partial \underline{U}_{s1}}{\partial t} + \epsilon^2 m_s n_{s0} \underline{U}_{s1} \cdot \nabla \underline{U}_{s1} + \epsilon^3 m_s n_{s1} \underline{U}_{s1} \cdot \nabla \underline{U}_{s1} \\
 & = \epsilon q_s n_{s0} \underline{E}_1 + \epsilon^2 q_s n_{s1} \underline{E}_1 + \epsilon n_{s0} q_s \underline{U}_{s1} \times \underline{B}_0 + \epsilon^2 n_{s1} q_s \underline{U}_{s1} \times \underline{B}_0 + \epsilon^3 n_{s0} q_s \underline{U}_{s1} \times \underline{B}_1 \\
 & \quad + \epsilon^2 n_{s0} q_s \underline{U}_{s1} \times \underline{B}_1
 \end{aligned}$$

d.  $\mathcal{O}(\epsilon)$  yields  $\boxed{m_s n_{s0} \frac{\partial \underline{U}_{s1}}{\partial t} = q_s n_{s0} \underline{E}_1 + q_s n_{s0} \underline{U}_{s1} \times \underline{B}_0}$

5. Fourier Transforming, dividing by  $m_s n_{s0}$ , and using  $\underline{B}_0 = B_0 \hat{z}$  give

$$\omega \underline{U}_{s1} = i \frac{q_s}{m_s} \underline{E}_1 + i \frac{q_s B_0}{m_s} \underline{U}_{s1} \times \hat{z}$$

6. Noting that  $\omega_{cs} \equiv \frac{q_s B_0}{m_s}$ , this gives the components:

$$\omega U_{sx1} = i \frac{q_s}{m_s} E_x + i \omega_{cs} U_{y1}$$

$$\omega U_{sy1} = i \frac{q_s}{m_s} E_y - i \omega_{cs} U_{x1}$$

$$\omega U_{sz1} = i \frac{q_s}{m_s} E_z$$

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## II. A (Continued)

7. This can be written as a matrix equation:

$$\begin{pmatrix} \omega & -i\omega c_s & 0 \\ +i\omega c_s & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix} \begin{pmatrix} U_{sx1} \\ U_{sy1} \\ U_{sz1} \end{pmatrix} = \frac{i q_s}{m_s} \begin{pmatrix} E_{x1} \\ E_{y1} \\ E_{z1} \end{pmatrix}$$

8. This can be inverted to give the solution of  $\underline{U}_s$  in terms of  $\underline{E}$ .

$$\begin{pmatrix} U_{sx1} \\ U_{sy1} \\ U_{sz1} \end{pmatrix} = \frac{q_s}{m_s} \begin{pmatrix} \frac{-i\omega}{\omega c_s^2 - \omega^2} & \frac{c_s}{\omega c_s^2 - \omega^2} & 0 \\ \frac{-\omega c_s}{\omega c_s^2 - \omega^2} & \frac{-i\omega}{\omega c_s^2 - \omega^2} & 0 \\ 0 & 0 & \frac{i}{\omega} \end{pmatrix} \begin{pmatrix} E_{x1} \\ E_{y1} \\ E_{z1} \end{pmatrix}$$

9. Now we can substitute in for  $\underline{U}_s$  in  $\underline{j} = \sum_s q_s n_{s0} \underline{U}_s$

$$\begin{pmatrix} j_x \\ j_y \\ j_z \end{pmatrix} = \sum_s \frac{n_{s0} q_s^2}{m_s} \begin{pmatrix} \frac{-i\omega}{\omega c_s^2 - \omega^2} & \frac{c_s}{\omega c_s^2 - \omega^2} & 0 \\ \frac{-\omega c_s}{\omega c_s^2 - \omega^2} & \frac{-i\omega}{\omega c_s^2 - \omega^2} & 0 \\ 0 & 0 & \frac{i}{\omega} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

10. Thus, we have found  $\underline{j} = \underline{\sigma} \cdot \underline{E}$

This is the conductivity tensor for a cold, magnetized plasma.

## B. The Plasma Dielectric Tensor:

1.  $\underline{\epsilon} = \underline{I} + \frac{i \underline{\sigma}}{\omega \epsilon_0}$

2. Using  $\omega p_s^2 \equiv \frac{n_{s0} q_s^2}{\epsilon_0 m_s}$ , we find the form

$$\underline{\epsilon} = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix}$$

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II. B2 (Continued)

where  $S \equiv 1 - \sum_s \frac{\omega p_s^2}{\omega^2 - \omega_{cs}^2}$

$$D \equiv \frac{\omega_{cs} \omega p_s^2}{\omega(\omega^2 - \omega_{cs}^2)}$$

and  $P = 1 - \sum_s \frac{\omega p_s^2}{\omega^2}$

B. The terms S & D stand for Sum & Difference. They can be written alternatively as

$$S = \frac{1}{2}(R+L) \quad \text{and} \quad D = \frac{1}{2}(R-L)$$

where  $R = 1 - \sum_s \frac{\omega p_s^2}{\omega(\omega + \omega_{cs})}$  ← Right-hand polarized mode

$L = 1 - \sum_s \frac{\omega p_s^2}{\omega(\omega - \omega_{cs})}$  ← Left-hand polarized mode

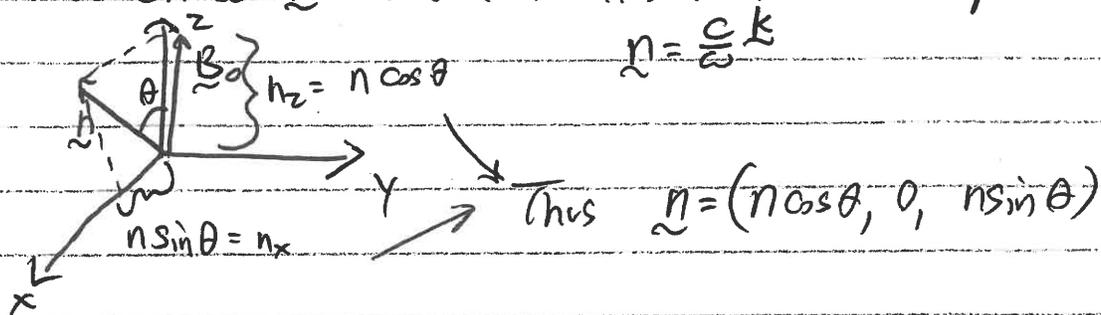
C. Dispersion Relation for a Cold, Magnetized Plasma

1. Remember, we want to solve the equation

$$\underline{n} \times (\underline{n} \times \underline{E}) + \underline{\epsilon} \cdot \underline{E} = 0$$

2. There are two special directions:  $\underline{B}_0$  and  $\underline{k}$ .

a. Let's choose  $\underline{k}$  so that it lies in the x-z plane.



b. We can show  $\underline{n} \times (\underline{n} \times \underline{E}) = (-n^2 \cos^2 \theta E_x + n^2 \sin \theta \cos \theta E_z) \hat{x} - n^2 E_y \hat{y} + (n^2 \sin \theta \cos \theta E_x - n^2 \sin^2 \theta E_z) \hat{z}$

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## II. C. (Continued)

3. Thus, our equation reduces to  $\underline{D} \cdot \underline{E} = 0$

$$\begin{pmatrix} S - n^2 \cos^2 \theta & -iD & n^2 \sin \theta \cos \theta \\ iD & S - n^2 & 0 \\ n^2 \sin \theta \cos \theta & 0 & P - n^2 \sin^2 \theta \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

The determinant  $|\underline{D}(n, \omega)| = 0$  is the dispersion relation.

4. This dispersion relation can be written in the form

$$\boxed{A n^4 - B n^2 + C = 0} \quad \text{"Booker Quartic"}$$

$$A = S \sin^2 \theta + P \cos^2 \theta$$

$$B = RL \sin^2 \theta + PS(1 + \cos^2 \theta)$$

$$C = RLP$$

a. This quadratic equation for  $n^2$  can be solved and put into the form:

$$n^2 = \frac{B \pm F}{2A}$$

$$\text{where } F^2 = (RL - PS)^2 \sin^4 \theta + 4P^2 D^2 \cos^2 \theta$$

b. Because  $F^2 > 0$ ,  $F$  must always be real.

Thus,  $n^2 > 0 \Rightarrow n$  is real  $\Rightarrow$  propagating wave

or  $n^2 < 0 \Rightarrow n$  is imaginary  $\Rightarrow$  evanescent wave

5. Alternative "Tangere" Form:

$$\boxed{\tan^2 \theta = \frac{-P(n^2 - R)(n^2 - L)}{(S n^2 - RL)(n^2 - P)}}$$