Lecture #13 Landau Damping of Electrostatic Waves

I. Laplace-Fourier Solution of Electrostatic Plasma Waves

A. Setup:

1. Electrostatic: $\underset{\sim}{E} = -\nabla\phi$, $\underset{\sim}{B} = 0$, $\underset{\sim}{E}_0 = 0 \Rightarrow \phi_0 = 0$
2. Vlasov-Poisson System:

$$\frac{\partial f_s}{\partial t} + \underset{\sim}{v}\cdot\nabla f_s - \frac{q_s}{m_s}\nabla\phi\cdot\frac{\partial f_s}{\partial \underset{\sim}{v}} = 0$$

$$-\nabla^2\phi = \frac{1}{\epsilon_0}\sum_s \int d^3\underset{\sim}{v}\, q_s f_s$$

3. Take $\underset{\sim}{k} = k\hat{\underset{\sim}{z}}$

B. Linearization

1. $$f_s = f_{s_0}(\underset{\sim}{v}) + \epsilon f_{s_1}(\underset{\sim}{x}, \underset{\sim}{v}, t)$$
$$\phi = \cancel{\phi_0}^{\,0} + \epsilon\phi_1(\underset{\sim}{x}, t)$$

2. At $\mathcal{O}(\epsilon)$: a. $$\frac{\partial f_{s_1}}{\partial t} + \underset{\sim}{v}\cdot\nabla f_{s_1} - \frac{q_s}{m_s}\nabla\phi_1\cdot\frac{\partial f_{s_0}}{\partial\underset{\sim}{v}} = 0$$

b. $$-\nabla^2\phi_1 = \frac{1}{\epsilon_0}\sum_s\int d^3\underset{\sim}{v}\, q_s f_{s_1}$$

C. Fourier Transform in Space Only $\nabla \Rightarrow i\underset{\sim}{k}$

1. a. $$\frac{\partial f_{s_1}}{\partial t} + i\underset{\sim}{v}\cdot\underset{\sim}{k} f_{s_1} - i\frac{q_s\phi_1}{m_s}\underset{\sim}{k}\cdot\frac{\partial f_{s_0}}{\partial\underset{\sim}{v}} = 0$$

b. $$k^2\phi_1 = \frac{1}{\epsilon_0}\sum_s\int d^3\underset{\sim}{v}\, q_s f_{s_1}$$

D. Laplace Transform in Time: $\tilde{f}_s(p) = \int_0^\infty dt\, f_{s_1}(t)\, e^{-pt}$

1. a. $$\tilde{f}_s'(p) + i\underset{\sim}{v}\cdot\underset{\sim}{k}\,\tilde{f}_s(p) - i\frac{q_s\tilde{\phi}_1(p)}{m_s}\underset{\sim}{k}\cdot\frac{\partial f_{s_0}}{\partial\underset{\sim}{v}} = 0$$

b. Using $\tilde{f}'(p) = p\tilde{f}(p) - f(0)$, we get

$$\boxed{(p + i\underset{\sim}{v}\cdot\underset{\sim}{k})\tilde{f}_s(p) = i\frac{q_s\tilde{\phi}_1(p)}{m_s}\underset{\sim}{k}\cdot\frac{\partial f_{s_0}}{\partial\underset{\sim}{v}} + f(0)}$$

I. D. (Continued)

2. Solving for $\tilde{f}_s(p)$

$$\boxed{\tilde{f}_s(p) = \frac{i\,\underset{\sim}{k}\cdot\frac{\partial f_{s0}}{\partial \underset{\sim}{v}}\,\frac{q_s\tilde{\phi}_1(p)}{m_s}}{p+i\underset{\sim}{k}\cdot\underset{\sim}{v}} + \frac{f_s(0)}{p+i\underset{\sim}{k}\cdot\underset{\sim}{v}}}$$

The poles in this solution are due to $\tilde{\phi}_1(p)$ poles and $p=-i\underset{\sim}{k}\cdot\underset{\sim}{v}$

E. Substitute $\tilde{f}_s(p)$ into Poisson's Equation to Solve for $\tilde{\phi}_1(p)$

1. $$k^2\tilde{\phi}_1 = \frac{1}{\epsilon_0}\sum_s\int d^3\underset{\sim}{v}\, q_s\left\{\frac{i\,\underset{\sim}{k}\cdot\frac{\partial f_{s0}}{\partial \underset{\sim}{v}}\,\frac{q_s\tilde{\phi}_1(p)}{m_s}}{p+i\underset{\sim}{k}\cdot\underset{\sim}{v}} + \frac{f_s(0)}{p+i\underset{\sim}{k}\cdot\underset{\sim}{v}}\right\}$$

NOTE: $\tilde{\phi}_1(p)$ does not depend on $\underset{\sim}{v}$.

2. Divide by k^2 and collect $\tilde{\phi}_1(p)$ terms:

a. $$\underbrace{\tilde{\phi}_1\left[1-\sum_s\frac{\left(\frac{q_s^2 n_0}{\epsilon_0 m_s}\right)}{k^2 n_0}\int d^3\underset{\sim}{v}\,\frac{i\underset{\sim}{k}\cdot\frac{\partial f_{s0}}{\partial\underset{\sim}{v}}}{p+i\underset{\sim}{k}\cdot\underset{\sim}{v}}\right]}_{\text{Dispersion Relation } D(p,\underset{\sim}{k})} = \underbrace{\frac{1}{k^2\epsilon_0}\sum_s\int d^3\underset{\sim}{v}\,\frac{q_s f_s(0)}{p+i\underset{\sim}{k}\cdot\underset{\sim}{v}}}_{\text{Initial Conditions } N(p,\underset{\sim}{k})}$$

b. Solution to $D(p,\underset{\sim}{k})=0$ gives normal modes of the system.

c. Thus $$\boxed{\tilde{\phi}_1(p) = \frac{N(p,\underset{\sim}{k})}{D(p,\underset{\sim}{k})}}$$

d. Inverse Laplace Transform of $\tilde{\phi}_1(p)$ by Residue Theorem is due to poles in $N(p,\underset{\sim}{k})$ and zeros of $D(p,\underset{\sim}{k})$

F. Simplify Using $\underset{\sim}{k}=k\hat{z}$ and Reduced Distribution Function $F_{s0}(v_z)$

1. $$F_{s0}(v_z) \equiv \frac{1}{n_0}\int_{-\infty}^{\infty}dv_x\int_{-\infty}^{\infty}dv_y\, f_{s0}(\underset{\sim}{v})$$

I. F. (Continued)

2. Thus a. $D(p,\underset{\sim}{k}) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2}\int_{-\infty}^{\infty} dv_z \underbrace{\frac{ik\frac{\partial F_{s0}}{\partial v_z}}{p+ikv_z}}_{=\frac{\partial F_{s0}/\partial v_z}{v_z - i\frac{p}{k}}} = 1 - \sum_s \frac{\omega_{ps}^2}{k^2}\int_{-\infty}^{\infty} dv_z \frac{\frac{\partial F_{s0}}{\partial v_z}}{v_z - \frac{ip}{k}}$

b Similarly

$$N(p,\underset{\sim}{k}) = \sum_s \frac{-iq_s n_0}{\epsilon_0 k^3}\int_{-\infty}^{\infty} dv_z \frac{F_s(0)}{v_z - \frac{ip}{k}}$$

3. Our Solution $\tilde{\phi}_1(\underset{\sim}{k},p)$ is then given by

Pieces of Solution due to:

$$\boxed{\tilde{\phi}_1(p,\underset{\sim}{k}) = \frac{-i\sum_s \frac{q_s n_0}{\epsilon_0 k^3}\int_{-\infty}^{\infty} dv_z \frac{F_s(0)}{v_z - ip/k}}{1 - \sum_s \frac{\omega_{ps}^2}{k^2}\int_{-\infty}^{\infty} dv_z \frac{\partial F_{s0}/\partial v_z}{v_z - ip/k}}}$$

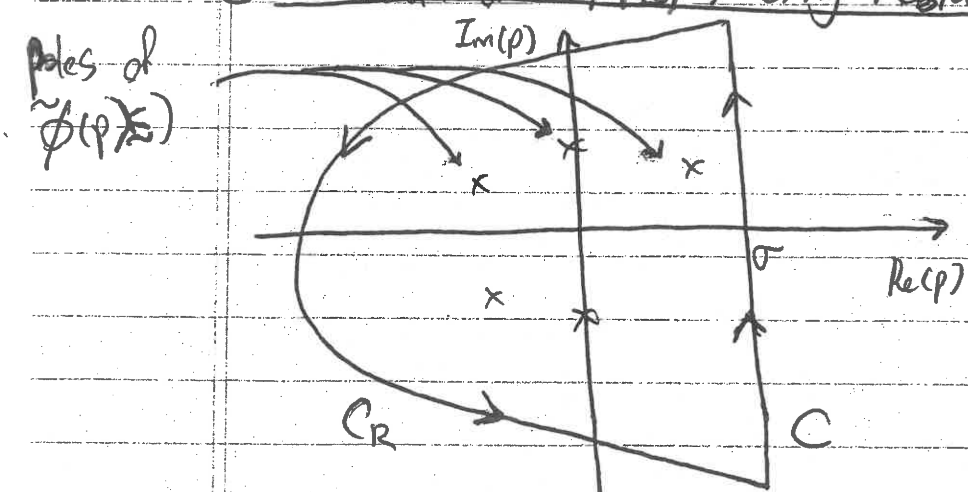
} Poles in Numerator

} Zeros in Denominator $D(p,\underset{\sim}{k}) = 0$ Normal Modes!

4. We want to find $\phi_1(\underset{\sim}{k},t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} dp\, \tilde{\phi}_1(\underset{\sim}{k},p)e^{pt}$

Using the Residue Theorem.

G. Evaluation of $\phi_1(\underset{\sim}{k},t)$ Using Residue Theorem



1. To Evaluate $\phi(\underset{\sim}{k},t)$ using the Residue Theorem, we close the contour by completing the loop at $Re(p)\to-\infty$ (This is section C_R)

Thus $\int_C dp\,\tilde{\phi}_1(\underset{\sim}{k},p)e^{pt} = \int_{\sigma-i\infty}^{\sigma+i\infty} dp\,\tilde{\phi}_1(\underset{\sim}{k},p)e^{pt} + \int_{C_R} dp\,\tilde{\phi}_1(\underset{\sim}{k},p)e^{pt}$

I. G. (Continued)

2. a. To evaluate contour integral using the Residue Theorem requires that $\tilde{\phi}_1(k,p)$ be analytic within and on contour C.

b. But, the function $\tilde{\phi}_1(k,p)$ was only defined for $Re(p) > 0$

$\Rightarrow$ Thus we must analytically continue $\tilde{\phi}_1(k,p)$ to the negative Real half plane $Re(p) < 0$.

c. This is not straight forward due to the v_z-integral in both $D(p,k)$ and $N(p,k)$. For example,

$$D(p,\underline{k}) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\partial F_{s0}/\partial v_z}{v_z - \frac{ip}{k}}$$

d. This function is discontinuous on the line $Re(p) = 0$.

Why? ① Remember $p = \gamma - i\omega$, so the denominator is

$$v_z - \frac{i}{k}(\gamma - i\omega) = v_z - \frac{\omega}{k} - \frac{i\gamma}{k}$$

② If $Re(p) = \gamma = 0$, then we have $\int_{-\infty}^{\infty} dv_z \frac{\partial F_{s0}/\partial v_z}{v_z - \frac{\omega}{k}}$

and the integral becomes undefined at $v_z = \frac{\omega}{k}$.

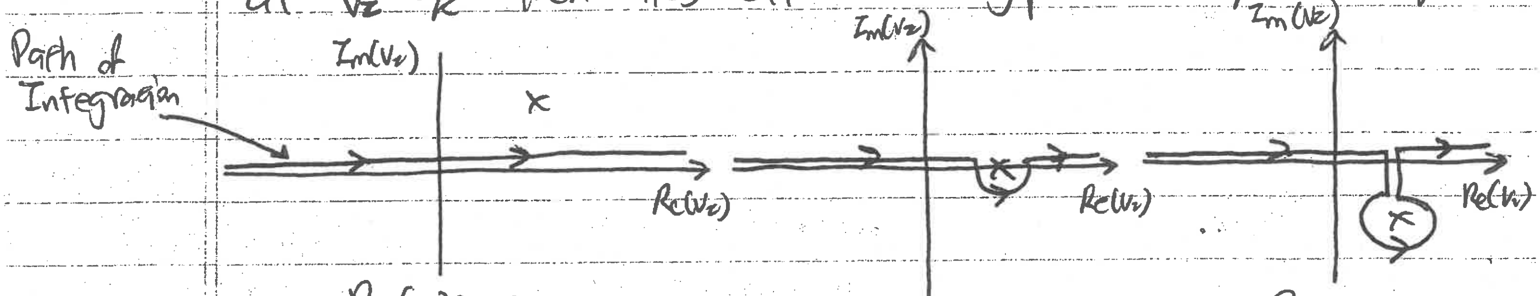
e. Since we must perform an contour integral over the entire complex plane p, this problem at $Re(p) = 0$ must be resolved.

H. Landau's Analytic Continuation of $D(p,\underline{k})$ and $N(p,\underline{k})$

1. Landau solved this problem by carrying out a careful analytic continuation of $D(p,\underline{k})$ and $N(p,\underline{k})$ to $Re(p) < 0$.

I. H. (Continued)

2. Consider the case $k>0$ ($k<0$ is analogous). The pole at $v_z = \frac{ip}{k}$ then lies at the following points in complex v_z space.



$Re(p) > 0$ $\quad$ $Re(p) = 0$ $\quad$ $Re(p) < 0$

a. Treating the integral $\int_{-\infty}^{\infty} dv_z$ as a contour integration in complex v_z space, Landau distorted the contour of integration so that it always passes below the pole in v_z space

b. In this way, the functions $D(p, \underline{k})$ and $N(p, \underline{k})$ [and thus $\tilde{\phi}_1(p, \underline{k})$] are analytically continued into the $Re(p)<0$ half of the complex p plane.

c. Now we can go ahead and use the Residue Theorem to evaluate $\int_{\sigma - i\infty}^{\sigma + i\infty} dp\, \tilde{\phi}_1(p, \underline{k}) e^{pt}$.

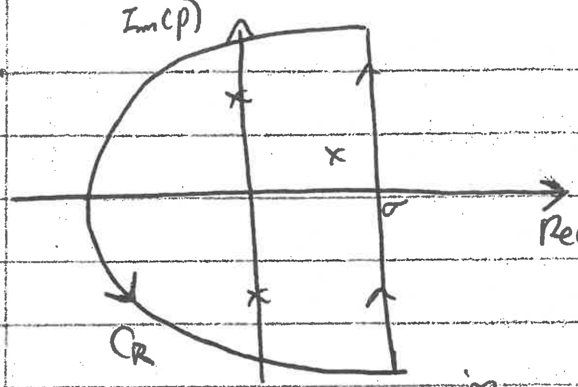
3. a. We'll look at concrete examples of this v_z integration soon.

b. For Maxwellian equilibrium distribution, this gives rise to the Plasma Dispersion Function.

I. Evaluation of $\phi_1(k, t)$

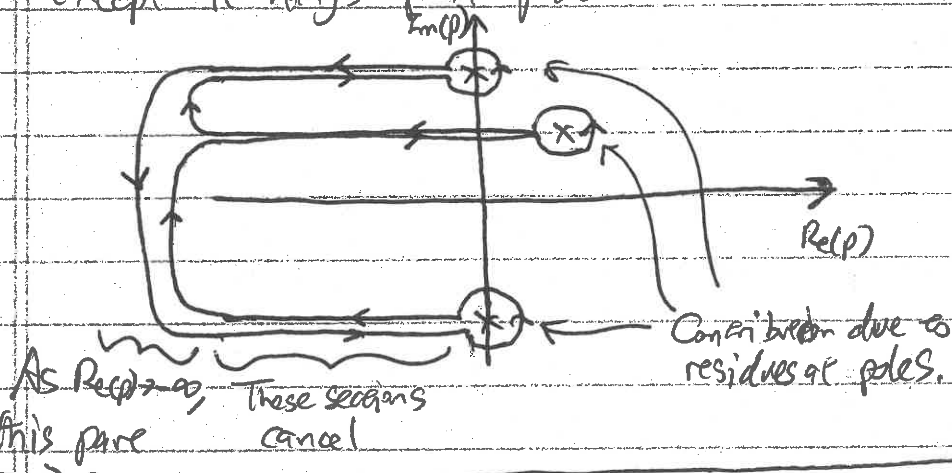
1. Remember $$f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} dp\, \tilde{f}(p) e^{pt}$$

I. I. (Continued)

2. 

a. $$\int_C dp\, \tilde{\phi}_1(k,p)e^{pt} = \underbrace{\int_{\sigma-i\infty}^{\sigma+i\infty} dp\, \tilde{\phi}_1(k,p)e^{pt}}_{=2\pi i\, \phi_1(\underline{k},t)} + \underbrace{\cancelto{0}{\int_{C_R} dp\, \tilde{\phi}_1(k,p)e^{pt}}}_{\text{As } Re(p)\to -\infty} = 2\pi i \sum_j \underset{p=p_j}{\text{Res}}\left[\tilde{\phi}_1(p)e^{pt}\right]$$

b. By <u>deformation of paths</u>, we can see that the only contribution to integral is due to residues. Deform C to $Re(p) \to -\infty$, except it hangs up at poles:



3. Thus, we find $$\boxed{\phi_1(\underline{k},t) = \sum_j \underset{p=p_j}{\text{Res}}\left[\tilde{\phi}_1(k,p)\, e^{pt}\right]}$$

4. Remember, p's are complex, $p = \gamma - i\omega$, so solutions typically have a behavior, $\sim e^{\gamma t} e^{-i\omega t}$, oscillatory with frequency ω and a growth rate for $\gamma > 0$, or damping rate for $\gamma < 0$.

II. Solution for Cauchy Velocity Distribution

A. Cauchy Velocity Distribution

1. A simple analytical distribution function is

<u>DEF:</u> <u>Cauchy Reduced Velocity Distribution</u> $F_{0_C}(v_z) = \frac{c}{\pi}\left(\frac{1}{c^2+v_z^2}\right)$

a. <u>NOTE:</u> $\int_{-\infty}^{\infty} dv_z\, F_{0_C}(v_z) = 1$

2. Consider ions immobile, so $F_{0i} = F_{0e}$ and $f_1 = 0$.

B. Velocity Integral over v_z

1. Our Dispersion Relation is $D(p,\underline{k}) = 1 - \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} dv_z \frac{\partial F_0/\partial v_z}{v_z - \frac{ip}{k}}$

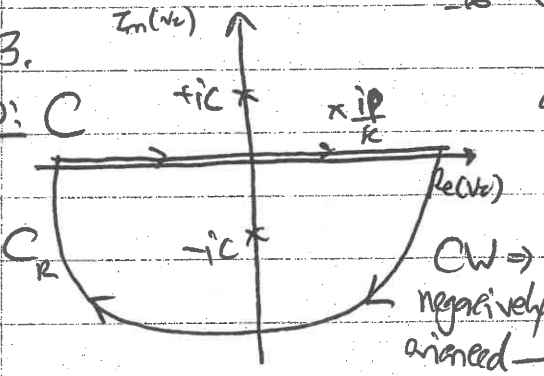
where we only consider the electron contribution since ions are immobile.

2. We can integrate by parts (as done in Lect #11, II. F. 3.) to yield

$$D(p,\underline{k}) = 1 - \frac{\omega_p^2}{k^2}\int_{-\infty}^{\infty} dv_z \frac{F_0}{(v_z - \frac{ip}{k})^2} = 1 - \frac{\omega_p^2}{k^2}\frac{c}{\pi}\int_{-\infty}^{\infty} dv_z \frac{1}{(v_z-ic)(v_z+ic)(v_z-\frac{ip}{k})^2}$$

3.

For $k>0$:



a. Close at $Im(v_z) \rightarrow -\infty$

b. Let $g(v_z) = \frac{1}{(v_z-ic)(v_z+ic)(v_z-\frac{ip}{k})^2}$

c. Thus $\int_C dv_z\, g(v_z) = \int_{-\infty}^{\infty} dv_z\, g(v_z) + \cancel{\int_{C_R} dv_z\, g(v_z)} \rightarrow 0$ as $Im(v_z) \rightarrow -\infty$ (Really $|v_z| \rightarrow \infty$)

$$= -2\pi i \sum_j \underset{v_z=v_{zj}}{Res}[g(v_z)]$$

d. Thus for pole at $v_z = -ic$

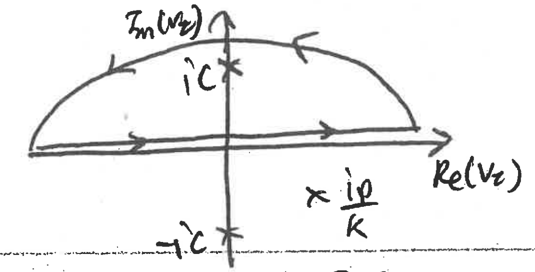
$$= -2\pi i \frac{1}{(-2ic)(-ic-\frac{ip}{k})^2} = \frac{\pi}{c}\frac{-1}{(c+\frac{p}{k})^2}$$

e. So, we find for $k>0$:

$$D(p,\underline{k}) = 1 + \frac{\omega_p^2 \cancel{c}}{k^2 \cancel{\pi}}\frac{\cancel{\pi}}{\cancel{c}}\frac{+1}{(c+\frac{p}{k})^2} = 1 + \frac{\omega_p^2}{(p+kc)^2}$$

Lecture # 13 (Continued)

II. B. (Continued).



4. Similarly for $k<0$

a. Close in upper half plane $Im(v_z) \to \infty$ (CCW orientation).

b. Thus $\int_{-\infty}^{\infty} dv_z\, g(v_z) = 2\pi i \sum_j \underset{v_z=v_{zj}}{\mathrm{Res}} [g(v_z)]$ → pole at $v_z = +iC$

$$= 2\pi i \frac{1}{2iC(iC - \frac{ip}{k})^2} = \frac{-\pi}{C} \frac{1}{(C-\frac{p}{k})^2}$$

c. Thus $D(p,\underline{k}) = 1 + \frac{\omega_p^2}{(p-kC)^2}$

5. a. Noting that for $k>0$, $k=|k|$ and for $k<0$ $k=-|k|$, we can write these as a single equation $\boxed{D(p,\underline{k}) = 1 + \frac{\omega_p^2}{(p+|k|C)^2} = 0}$

b. NOTE: Since this solution is a polynomial, analytic continuation to the $Re(p)<0$ plane is trivial.

6. Roots of dispersion relation are $\boxed{p = -|k|C \pm i\omega_p}$

C. Solving for $N(\underline{k},p)$

Initial condition on F_{s_1}

1. $N(\underline{k},p) = -i \sum_s \frac{q_s n_0}{\epsilon_0 k^3} \int_{-\infty}^{\infty} dv_z \frac{F_s(\underline{k},\underline{v},0)}{v_z + ip/k}$

a. If we have a specific form for the initial conditions $F_s(\underline{k},\underline{v},0)$, then we can perform the integral analogous to the procedure above.

b. An important point is that, as long as $F_s(\underline{k},\underline{v},0)$ does not have any singularities or discontinuities, the result of the integration will not have any singularities. → Thus, no poles in $N(\underline{k},p)$

II. C. (Continued)

2. Rather than solve for a specific form of $F_s(\underset{\sim}{k}, \underset{\sim}{v}, 0)$, we note

$$\tilde{\phi}(\underset{\sim}{k}, p) \underbrace{D(\underset{\sim}{k}, p)}_{\text{Dispersion Relation}} = \underbrace{N(\underset{\sim}{k}, p)}_{\text{Initial Conditions}}$$ (see I.E.2.a. earlier)

a. We simply denote $N(\underset{\sim}{k}, p) = \frac{1}{\omega_p}\phi(\underset{\sim}{k}, 0)$ since it is determined by the initial conditions.

b. Thus $$\tilde{\phi}(k, p) = \frac{\phi(\underset{\sim}{k}, 0)}{\omega_p D(p, \underset{\sim}{k})} = \frac{\phi(\underset{\sim}{k}, 0)}{\omega_p\left(1 + \frac{\omega_p^2}{(p+|k|C)^2}\right)} = \frac{(p+|k|C)^2\phi(\underset{\sim}{k}, 0)}{[(p+|k|C)^2 + \omega_p^2]\omega_p}$$

D. Completing Solution for $\phi(\underset{\sim}{k}, t)$

1. As we solved earlier (I. I, 3.), $\phi(\underset{\sim}{k}, t) = \sum_j \underset{p=p_j}{\text{Res}}\left[\tilde{\phi}(\underset{\sim}{k}, p)e^{pt}\right]$

a. Here $$\tilde{\phi}(\underset{\sim}{k}, p)e^{pt} = \frac{(p+|k|C)^2\phi(\underset{\sim}{k}, 0)e^{pt}}{(p+|k|C-i\omega_p)(p+|k|C+i\omega_p)\omega_p}$$

Poles at roots $p = -|k|C + i\omega_p$ & $p = -|k|C - i\omega_p$

2. Thus $$\phi(k, t) = \frac{(\cancel{-|k|C}+i\omega_p \cancel{+|k|C})^2\phi(k, 0)e^{-|k|Ct}e^{i\omega_p t}}{(\cancel{-|k|C}+i\omega_p \cancel{+|k|C}+i\omega_p)\omega_p} + \frac{(\cancel{-|k|C}-i\omega_p \cancel{+|k|C})^2\phi(k, 0)e^{-|k|Ct}e^{-i\omega_p t}}{(\cancel{-|k|C}-i\omega_p \cancel{+|k|C}-i\omega_p)\omega_p}$$

$$= \frac{-\cancel{\omega_p^2}\phi(\underset{\sim}{k}, 0)e^{-|k|Ct}e^{i\omega_p t}}{2i\cancel{\omega_p^2}} + \frac{-\cancel{\omega_p^2}\phi(\underset{\sim}{k}, 0)e^{-|k|Ct}e^{-i\omega_p t}}{-2i\cancel{\omega_p^2}}$$

$$\boxed{\phi(\underset{\sim}{k}, t) = -\phi(\underset{\sim}{k}, 0)e^{-|k|Ct}\left(\frac{e^{i\omega_p t} - e^{-i\omega_p t}}{2i}\right)} = -\phi(\underset{\sim}{k}, 0)\sin(\omega_p t)e^{-|k|Ct}$$