Lecture #14 Maxwellian Distributions and the Plasma Dispersion Function

I. Review

A. Laplace-Fourier Solution of Electrostatic Waves: Langmuir Waves & Landau Damping

1. The Laplace-Fourier transformed solutions for $\phi_1(\underline{x},t) \Rightarrow \tilde{\phi}(\underline{k},p)$

$$\tilde{\phi}(\underline{k},p) = \frac{N(\underline{k},p)}{D(\underline{k},p)}$$

where the numerator is $N(\underline{k},p) = -i \sum_s \frac{q_s n_0}{\epsilon_0 k^3} \int_{-\infty}^{\infty} dv_z \frac{F_s(\underline{k},0)}{v_z - ip/k}$

and the denominator is the dispersion relation

$$\boxed{D(\underline{k},p) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\partial F_{s0}/\partial v_z}{v_z - ip/k}}$$

2. Often, we only care about the properties of the wave, such as the real frequency ω and damping (or growth) rate γ, rather than requiring a full solution of $\phi_1(\underline{x},t)$ and $f_{s1}(\underline{x},\underline{v},t)$.

3. For the Cauchy velocity distribution $F_{0C}(v_z) = \frac{C}{\pi} \frac{1}{C^2+v_z^2}$, we found $p = -|k|C \pm i\omega_p$. Since $p = \gamma - i\omega$ (where γ & ω are real), we get

$$\boxed{\begin{array}{ll} \text{Real Frequency} & \omega = \pm\omega_p \\ \text{Damping Rate} & \gamma = -|k|C \end{array}}$$

4. The damping is called Landau Damping, first discovered by Landau in this formal mathematical solution in 1946.

II. Weak Growth Rate Approximation:

A. Overview

1. We can learn more about the general properties of Landau damping using a powerful approach in the limit of weak growth $|\gamma| \ll |\omega|$.

2. Use of Approximations and Limits
   a. A general kinetic treatment of most plasma systems of interest leads to a set of equations that are analytically intractable.
   b. In this case, one may choose to solve the system numerically.
   c. But, when you obtain the numerical solution, how do we know that it is correct?
   d. One approach to validate numerically results is to solve the system again numerically with an independent method. This often means doing the problem twice.
   e. Another means to validate numerical results is to solve for the system in limits that yield to an analytical approach.
   f. If the numerical solution agrees in the various analytical limits, then one may have confidence in the numerical solution.

THE IMPORTANCE OF VALIDATING NUMERICAL RESULTS CANNOT BE UNDERESTIMATED

B. Taylor Expansion about $p=-i\omega$ for $|\gamma| \ll |\omega|$

1. Consider the (complex) dispersion relation $D(\underline{k}, p)$ where $p = \gamma - i\omega$ (we treat γ and ω as real).
2. For $|\gamma| \ll |\omega|$, let's Taylor Expand the dispersion relation $D(\underline{k}, p)$ about the point $p=-i\omega$.
   a. $f(x) = f(x_0) + (x-x_0)\dfrac{\partial f(x_0)}{\partial x} + \dfrac{(x-x_0)^2}{2}\dfrac{\partial^2 f(x_0)}{\partial x^2} + \ldots$

      where $x_0 = -i\omega$ $x = \gamma - i\omega = p$, so $x - x_0 = \gamma$

Drop $\underline{k}$ dependence → it is assumed.

   b. Thus $D(p) = D(-i\omega) + \gamma \dfrac{\partial D(-i\omega)}{\partial p} = D(-i\omega) + \gamma \dfrac{\partial \omega}{\partial p}\dfrac{\partial D(-i\omega)}{\partial \omega} + \gamma \dfrac{\partial \gamma}{\partial p}\dfrac{\partial D(-i\omega)}{\partial \gamma}$

II. B2 (Continued)

c. Noting that $\frac{\partial \omega}{\partial p} = i$ and $\frac{\partial \gamma}{\partial p} = 1$, we also note that

$$\frac{\partial D(-i\omega)}{\partial \gamma} = 0$$ since $D(-i\omega)$ does not depend on γ.

d. Thus,
$$D(p) = D(-i\omega) + i\gamma \frac{\partial D(-i\omega)}{\partial \omega}$$

3. Since $D(\underset{\sim}{k}, p)$ is complex, we write $D(\underset{\sim}{k},-i\omega) = D_r(\underset{\sim}{k},-i\omega) + i D_i(\underset{\sim}{k},-i\omega)$

a. Substituting, we find
$$D(\underset{\sim}{k},p) = D_r(\underset{\sim}{k},-i\omega) + i D_i(\underset{\sim}{k},-i\omega) + i\gamma \frac{\partial D_r(\underset{\sim}{k},-i\omega)}{\partial \omega} - \gamma \frac{\partial D_i(\underset{\sim}{k},-i\omega)}{\partial \omega} = 0$$

4. To Lowest order, we find:

Weak Growth Rate Approximation

a. $\boxed{D_r(\underset{\sim}{k},-i\omega) = 0}$ ⇐ From real part $=0$ (Note $\gamma \ll \omega$)

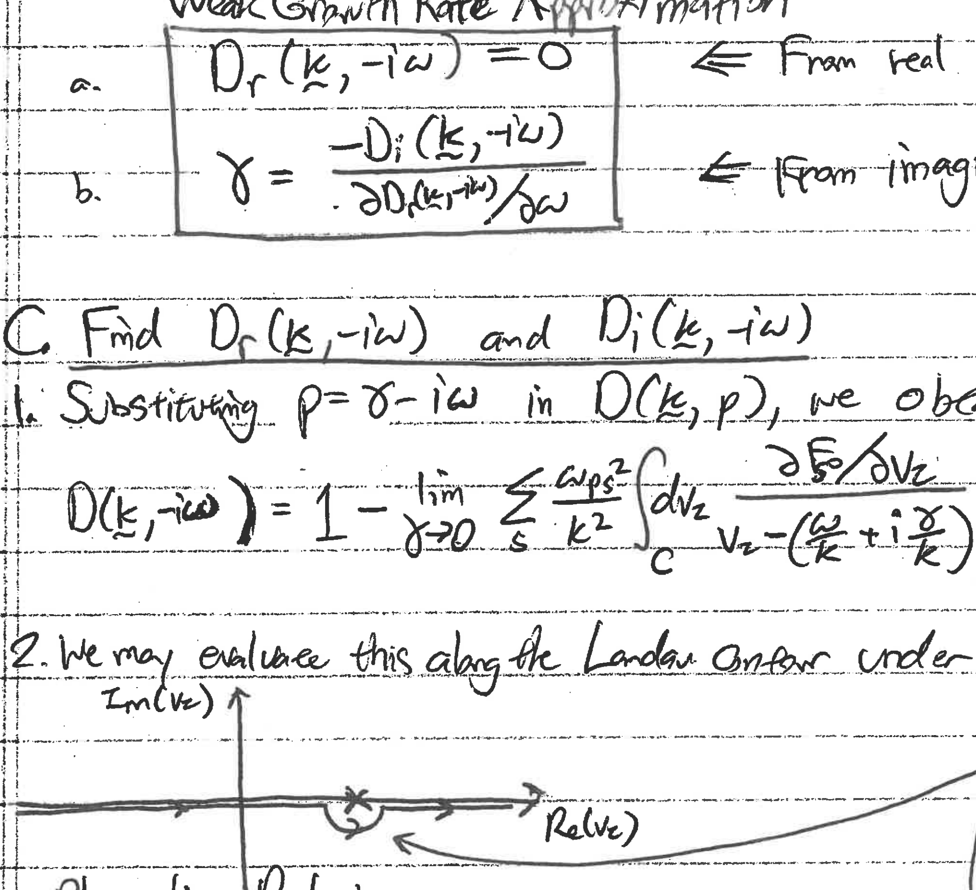
b. $\boxed{\gamma = \frac{-D_i(\underset{\sim}{k},-i\omega)}{\partial D_r(\underset{\sim}{k},-i\omega)/\partial\omega}}$ ⇐ From imaginary part $=0$

C. Find $D_r(\underset{\sim}{k},-i\omega)$ and $D_i(\underset{\sim}{k},-i\omega)$

1. Substituting $p = \gamma - i\omega$ in $D(\underset{\sim}{k},p)$, we obtain

$$D(\underset{\sim}{k},-i\omega) = 1 - \lim_{\gamma\to 0} \sum_s \frac{\omega_{ps}^2}{k^2} \int_C dv_z \frac{\partial F_{s0}/\partial v_z}{v_z - \left(\frac{\omega}{k} + i\frac{\gamma}{k}\right)}$$

2. We may evaluate this along the Landau contour under the pole



Only halfway around pole at $Im(v_z) = \frac{\gamma}{k} = 0$.

3. Plemelj Relation:

a. $$\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} dx \frac{f(x)}{x-(x_0 \pm i\epsilon)} = P\int_{-\infty}^{\infty} dx \frac{f(x)}{x-x_0} \pm i\pi f(x_0)$$

II. C. 3. (Continued)

b. Here, the Principal Value of the integral, denoted by P is given by

Principal Value: $$P\int_{-\infty}^{\infty} \frac{g(x)}{x-x_0}\,dx \equiv \lim_{\delta\to 0}\left[\int_{-\infty}^{x_0-\delta} dx\,\frac{g(x)}{x-x_0} + \int_{x_0+\delta}^{\infty} dx\,\frac{g(x)}{x-x_0}\right]$$

4. Applying the Plemelj ~~Relation~~, we find

$$\boxed{\begin{aligned} D_r(\underset{\sim}{k}, -i\omega) &= 1 - \sum_s \frac{\omega_{ps}^2}{k^2} P\int_{-\infty}^{\infty} dv_z \frac{\partial F_{s0}/\partial v_z}{v_z - \frac{\omega}{k}} \\ D_i(\underset{\sim}{k}, -i\omega) &= -\pi \frac{k}{|k|} \sum_s \frac{\omega_{ps}^2}{k^2} \left.\frac{\partial F_{s0}}{\partial v_z}\right|_{v_z=\frac{\omega}{k}} \end{aligned}}$$

a. NOTE that the Principal Value Integral provides a formal mathematical procedure for avoiding divergence of the integral at $v_z = \frac{\omega}{k}$.

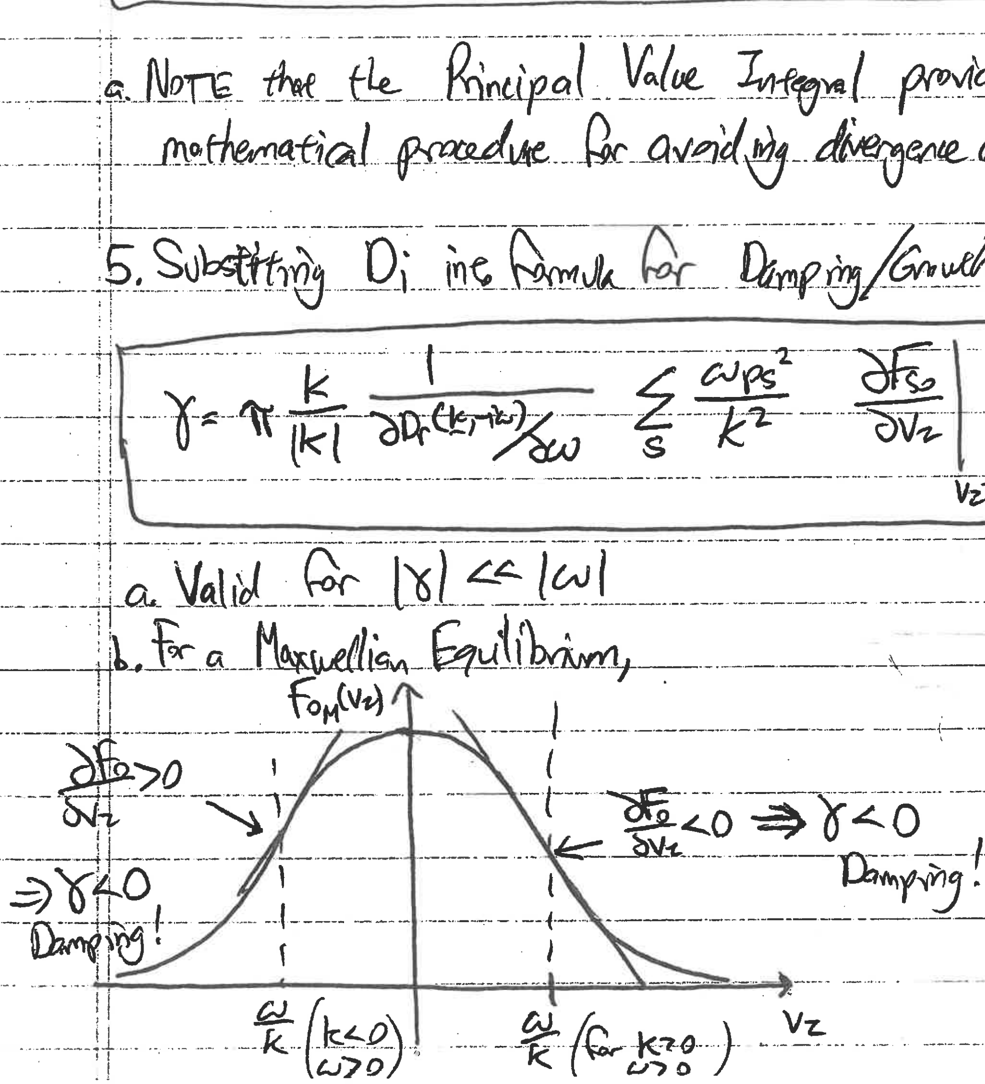
5. Substituting D_i into formula for Damping/Growth rate γ,

$$\boxed{\gamma = \pi \frac{k}{|k|} \frac{1}{\partial D_r(k,-i\omega)/\partial\omega} \sum_s \frac{\omega_{ps}^2}{k^2} \left.\frac{\partial F_{s0}}{\partial v_z}\right|_{v_z=\frac{\omega}{k}}}$$

Weak Growth Rate Approximation for Electrostatic (Langmuir) Waves

a. Valid for $|\gamma| \ll |\omega|$

b. For a Maxwellian Equilibrium,



NOTE: Typically, $\mathrm{sgn}\left(\frac{\partial D_r}{\partial\omega}\right) = \mathrm{sgn}(\omega)$

II. C. 5. (Continued)

c. Growth/Damping Rate is proportional to slope of equilibrium distribution function at wave phase velocity $v_z = \frac{\omega}{k}$.

d. For a single maximum in $F_0(v_z)$ at $v_z = 0$, we always have damping $\gamma < 0$. (More on this next).

III. Bohm-Gross Dispersion Relation for Maxwellian Plasma

A. Result of Laplace-Fourier Solution

1. In the high phase velocity limit $v_{te} \ll \frac{\omega}{k}$, the real part of the dispersion relation in the same as that derived using the Fourier approach (See Lect #11, II.F.6)

$$D_r(k,\omega) = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2\langle v_z^2\rangle}{\omega^2}\right) = 0$$ (assuming immobile ions)

2. To lowest order in ω, we can solve for $\frac{\partial D_r}{\partial \omega}$ to find γ:

a. $$\frac{\partial D_r}{\partial \omega} = \frac{2\omega_p^2}{\omega^3} + \underbrace{\frac{12\omega_p^2 k^2\langle v_z^2\rangle}{\omega^5}}_{12\frac{\omega_p^2}{\omega^3}\underbrace{\left(\frac{k^2\langle v_z^2\rangle}{\omega^2}\right)}_{} \ll \frac{2\omega_p^2}{\omega^3}} \simeq \frac{2\omega_p^2}{\omega^3}$$

$$\frac{k v_{te}}{\omega} \ll 1$$

b. Lowest order solution for $\omega = \omega_p$, so

$$\left.\frac{\partial D_r}{\partial \omega}\right|_{\omega=\omega_p} = \frac{2}{\omega_p}$$

c. Thus $$\boxed{\gamma = \frac{\pi}{2}\frac{k}{|k|}\frac{\omega_p^3}{k^2}\left.\frac{\partial F_0}{\partial v_z}\right|_{v_z=\frac{\omega_p}{k}}}$$

As expected, growth rate depends of slope of equilibrium velocity distribution.

3. For a Maxwellian Distribution, $F_e(v_z) = \frac{e^{-v_z^2/v_{te}^2}}{\pi^{1/2} v_{te}}$

a. As before (Lect #11 II.F.9) $\langle v_z^2\rangle = \frac{T_e}{m_e} = \frac{v_{te}^2}{2}$

III. A. 3. (Continued)

b. Thus, $\boxed{\omega^2 = \omega_{pe}^2 + \frac{3}{2} k^2 v_{te}^2}$ Real Frequency, Langmuir Waves

c. $$\frac{\partial f_{0e}}{\partial v_z} = -\frac{2 v_z}{v_{te}^3} \frac{e^{-\frac{v_z^2}{v_{te}^2}}}{\pi^{1/2}}$$

We then substitute $v_z = \frac{\omega}{k}$

NOTES: In exponent, we keep the small term $v_z^2 = \frac{\omega^2}{k^2} = \frac{\omega_{pe}^2}{k^2} + \frac{3}{2} v_{te}^2$.

Thus $$\frac{\partial f_{0e}}{\partial v_z} = \frac{-2\omega_{pe}}{k v_{te}^3} \frac{e^{-\frac{\omega_{pe}^2}{k^2 v_{te}^2} - \frac{3}{2}}}{\pi^{1/2}}$$

d. Noting that $\frac{\omega_{pe}^2}{v_{te}^2} = \frac{1}{2\lambda_{De}^2}$, we find

$$\boxed{\gamma = -\sqrt{\frac{\pi}{8}} \frac{\omega_p}{|k|^3 \lambda_{De}^3} e^{-\frac{1}{2k^2\lambda_{De}^2} - \frac{3}{2}}}$$ Landau Damping, Langmuir Waves.

4. a. The real frequency is the same as the Bohm-Gross dispersion relation derived using the Fourier method.

b. The damping rate, which is always negative, is weak at large wavelength $k\lambda_{De} \ll 1$, but becomes strong as $k\lambda_{De} \to 1$.

IV. The Plasma Dispersion Function (NRL p. 30)

A. 1. The dispersion relation for a Maxwellian velocity distribution can be nicely expressed in terms of the Plasma Dispersion Function

2. Consider $$D(\underset{\sim}{k}, p) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\partial f_{s0}/\partial v_z}{v_z - ip/k}$$ where $$\frac{\partial f_{s0}}{\partial v_z} = \frac{-2 v_z}{v_{ts}^3} \frac{e^{-\frac{v_z^2}{v_{ts}^2}}}{\pi^{1/2}}$$

$$= 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \frac{(-2)}{v_{ts}^2 \pi^{1/2}} \int \frac{dv_z}{v_{ts}} \frac{v_z}{v_{ts}} \frac{e^{-\frac{v_z^2}{v_{ts}^2}}}{\frac{v_z}{v_{ts}} - \frac{ip}{k v_{ts}}}$$

IV. A. (Continued)

3. a. Define: $z \equiv \frac{v_z}{v_{ts}}$ and $\xi_s \equiv \frac{ip}{k v_{ts}}$

b. Thus

$$D(\underset{\sim}{k}, p) = 1 + \sum_s \frac{1}{k^2 \lambda_{Ds}^2} \int_{-\infty}^{\infty} \frac{dz}{\pi^{1/2}} \frac{z e^{-z^2}}{z - \xi_s}$$

4. a. Manipulate $\frac{z}{z-\xi_s} = \frac{z-\xi_s+\xi_s}{z-\xi_s} = 1 + \frac{\xi_s}{z-\xi_s}$

b. Since $\int_{-\infty}^{\infty} \frac{dz}{\pi^{1/2}} e^{-z^2} = 1$, we get $D(\underset{\sim}{k}, p) = 1 + \sum_s \frac{1}{k^2 \lambda_{Ds}^2}\left[1 + \xi_s Z(\xi_s)\right]$

where we have defined

DEF: The Plasma Dispersion Function: $\boxed{Z(\xi_s) = \int_C \frac{dz}{\pi^{1/2}} \frac{e^{-z^2}}{z-\xi_s}}$

Ref: (Fried & Conte, 1961)

where the integral is over the Landau contour below the pole from $-\infty$ to ∞.

5. For a plasma with immobile ions, we have

$$\boxed{D(k, p) = 1 + \frac{1}{k^2 \lambda_{De}^2}\left[1 + \xi_e Z(\xi_e)\right] = 0}$$

6. NOTE: Since $p = \gamma - i\omega$, $\boxed{\xi_s = \frac{\omega}{k v_{ts}} + i \frac{\gamma}{k v_{ts}}}$

B. Power Series Expansions of $Z(\xi)$

1. By expressing the dispersion relation in terms of the plasma dispersion function, one can achieve a nice concise form.

2. Expansions of $Z(\xi)$ in various limits enable the determination of limiting behaviors quite easily. (see Homework)

Lecture #14 (Continued)

IV. B. (Continued)

3. Large Argument Expansion: $|\xi| \gg 1$

a. This is the cold limit $\frac{\omega}{k} \gg v_t$

b. $$\boxed{Z(\xi) \simeq i\pi^{\frac{1}{2}} e^{-\xi^2} - \frac{1}{\xi} - \frac{1}{2\xi^3} - \frac{3}{4\xi^5} - \dots}$$ (NRL p.30)

4. Small Argument Expansion: $|\xi| \ll 1$

a. Low frequency or small wavelength limit, $\frac{\omega}{k} \ll v_t$

b. $$\boxed{Z(\xi) \simeq i\pi^{\frac{1}{2}} e^{-\xi^2} - 2\xi + \frac{4\xi^3}{3} - \frac{8\xi^5}{15} + \dots}$$ (NRL p.30)

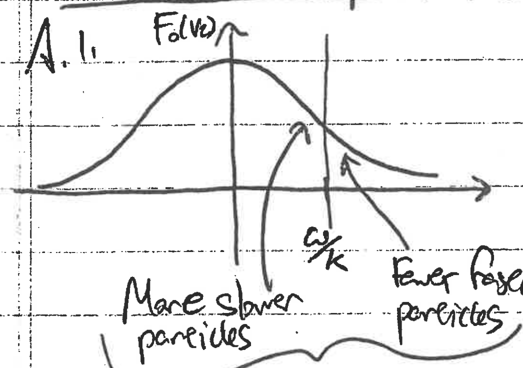
5. Example: Using the $|\xi| \gg 1$ limit, it can easily be shown

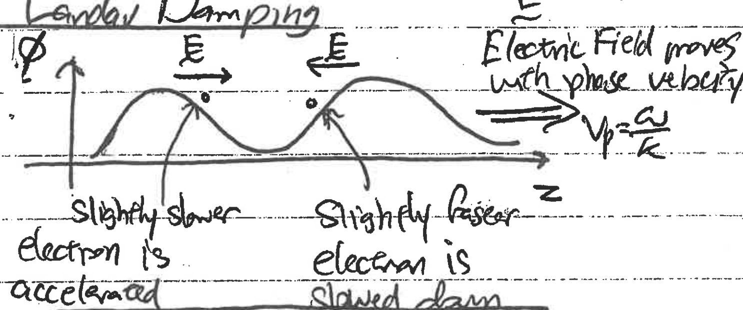
$$\omega^2 = \omega_{pe}^2 + \frac{3}{2} k^2 v_{te}^2 \quad \text{and} \quad \gamma = -\sqrt{\frac{\pi}{8}} \frac{\omega_p}{|k|^3 \lambda_{De}^3} e^{-\frac{1}{2k^2\lambda_{De}^2} - \frac{3}{2}}$$

V. Physical Interpretation of Landau Damping

$E = -\nabla\phi$

A. 1.

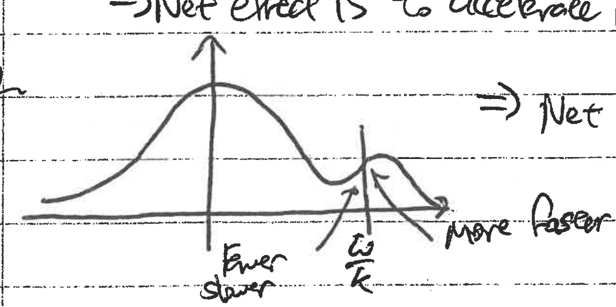




$\boxed{\text{Wave-Particle Interaction}}$

$\Rightarrow$ Net effect is to accelerate particles $\Rightarrow$ damps the wave

2.



$\Rightarrow$ Net effect is to slow down particles

$\Rightarrow$ gives energy to wave

3.