I. Review: Landau Damping of Langmuir Waves

A. Electrostatic Approximation: Vlasov-Poisson System

1. $\dfrac{\partial f_s}{\partial t} + \underset{\sim}{v} \cdot \nabla f_s - \dfrac{q_s}{m_s} \nabla \phi \cdot \dfrac{\partial f_s}{\partial \underset{\sim}{v}} = 0$

2. $-\nabla^2 \phi = \dfrac{1}{\epsilon_0} \sum_s \int d^3\underset{\sim}{v}\, q_s f_s$

B. Laplace Transform Approach by Landau

1. Linearization, Fourier Transform in Space, Laplace Transform in Time

$$\tilde{\phi}_1(\underset{\sim}{k}, p) = \frac{N(k,p)}{D(k,p)} \qquad \text{where } N(\underset{\sim}{k},p) = -i \sum_s \frac{q_s n_0}{\epsilon_0 k^3} \int_{-\infty}^{\infty} dv_z \frac{F_s(v)}{v_z - \frac{ip}{k}}$$
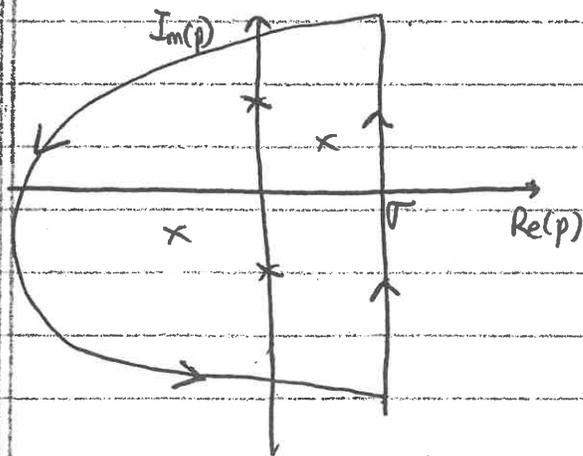
and the __Dispersion Relation__ is $D(k,p) = 0$ where

$$\boxed{D(k,p) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \frac{\partial F_{s0}/\partial v_z}{v_z - \frac{ip}{k}}}$$

$D(k,p) = 0$ gives normal modes of system.

2. To solve, the Inverse Laplace Transform use

a. $\phi_1(\underset{\sim}{k}, t) = \dfrac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} dp\, \tilde{\phi}_1(\underset{\sim}{k}, p) e^{pt}$

b. We evaluate this complex contour integral by closing the contour at $\text{Re}(p) \to -\infty$ and using the residue theorem.

c. To do so, we must analytically continue $\tilde{\phi}_1(\underset{\sim}{k}, p)$ from $\text{Re}(p) > 0$ to $\text{Re}(p) < 0$.
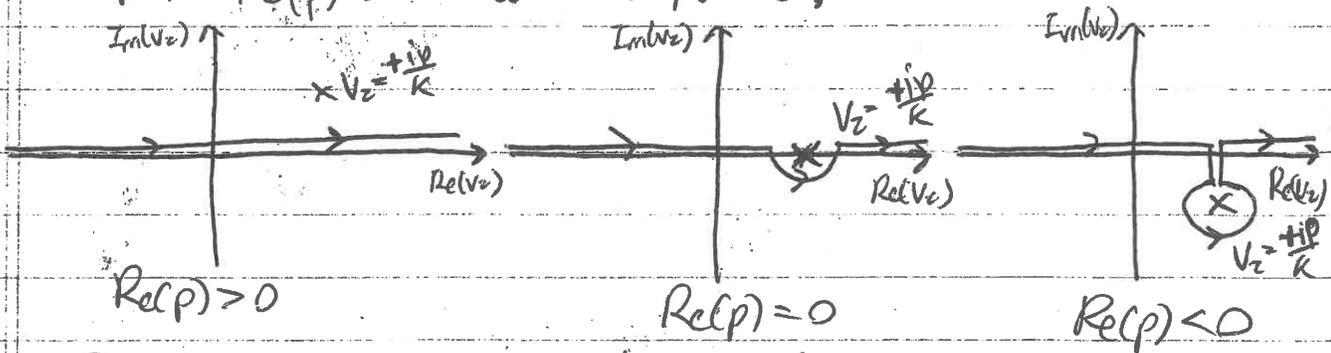
Lecture #15 (Continued)

I. A. (Continued)

3. How do we analytically continue $D(\underline{k}, p) = 1 - \sum\limits_{s} \frac{\omega_{ps}^2}{k^2} \int\limits_{-\infty}^{\infty} dv_z \frac{\partial f_{so}/\partial v_z}{v_z - \frac{ip}{k}}$
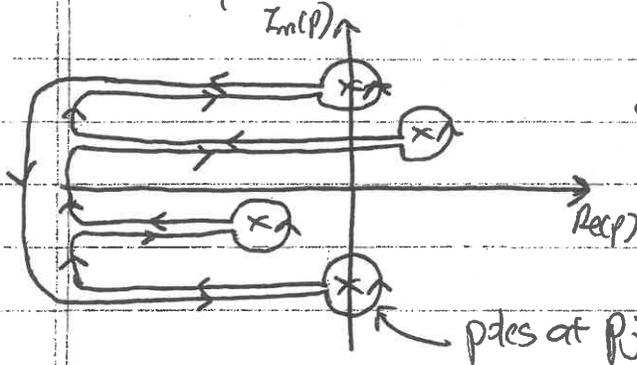
from $Re(p) > 0$ to $Re(p) < 0$?

$Re(p) > 0$ $Re(p) = 0$ $Re(p) < 0$

$\Rightarrow$ Contour always passes below pole !

This definition yields an analytic form of $\tilde{\phi}(k, p)$ over entire complex p-plane (except for poles, of course).

4. We may then evaluate the Inverse Laplace Transform by Residue Theorem

poles at p_j

$$a. \quad \phi_1(\underline{k}, t) = \sum\limits_{j} \operatorname*{Res}_{p=p_j} \left[\tilde{\phi}_1(\underline{k}, p) e^{pt} \right]$$

C. Landau Damping of Waves

1. For a Cauchy Distribution of electrons with stationary ions,

a. $F_{0_c}(v_z) = \frac{C}{\pi} \left(\frac{1}{C^2 + v_z^2} \right)$

b. Dispersion Relation: $D(\underline{k}, p) = 1 + \frac{\omega_p^2}{(p + |k| C)^2} = 0$

c. Wave Solutions with $\omega = \pm \omega_p$ and $\gamma = -|k| C$

       ↖ Landau damping due to
       $Re(p) < 0$ (or $Im(\omega) < 0$)

Lecture #5 (Continued)

I.C. (Continued)

2. Maxwellian Distribution:

a. The Plasma Dispersion Function

$$Z(\xi_s) = \int_C \frac{dz}{\pi^{1/2}} \frac{e^{-\xi_s^2}}{z - \xi_s}$$

where $\xi_s = \frac{ip}{kv_{es}}$ or $\boxed{\xi_s = \frac{\omega}{kv_{es}} + i\frac{\gamma}{kv_{es}}}$

b. Using this function, the Langmuir Wave Dispersion Relation can be written,

$$D(k,p) = 1 + \frac{1}{k^2\lambda_{De}^2}\left[1 + \xi_e Z(\xi_e)\right] = 0$$

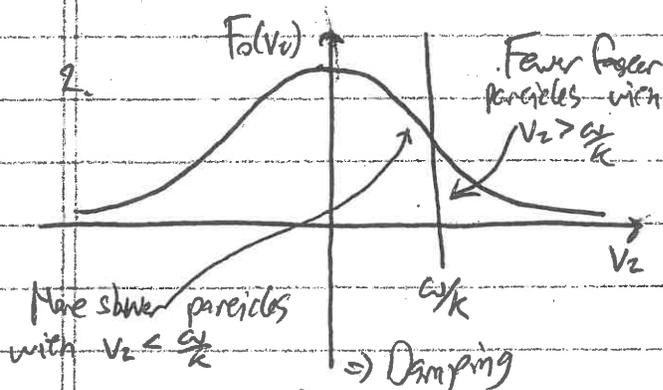c. In the high phase velocity limit, $|\xi_e| \gg 1$, or $\frac{\omega}{k} \gg v_{Te}$, we find

$$\omega^2 = \omega_{pe}^2 + \frac{3}{2}k^2 v_{Te}^2 \qquad \text{and} \qquad \gamma = -\sqrt{\frac{\pi}{8}}\frac{\omega_{pe}}{|k|^3\lambda_{De}^3}e^{\left(\frac{-1}{2k^2\lambda_{De}^2} - \frac{3}{2}\right)}$$
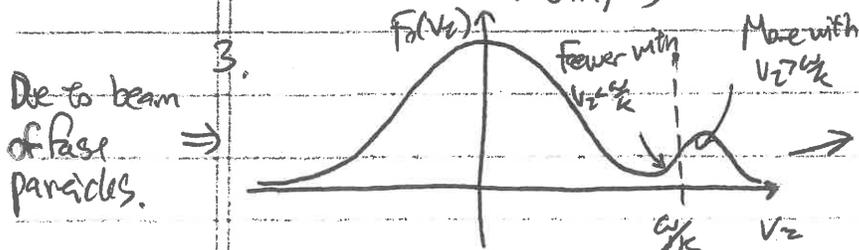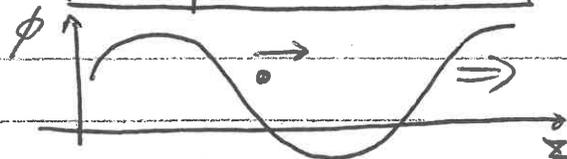
↖ Landau Damping.

D. Physical Interpretation of Landau Damping

1. In the Weak Growth Rate Approximation, $|\gamma| \ll |\omega|$,

$$\gamma = \pi \frac{k}{|k|}\frac{1}{\partial D_r/\partial\omega}\sum_s \frac{\omega_{ps}^2}{k^2}\frac{\partial F_{s0}}{\partial v_z}\bigg|_{v_z = \frac{\omega}{k}}$$

↖ Damping is due to Resonant particles in distribution func.

$\boxed{\text{Wave-particle Interaction}}$

2.

$F_0(v_z)$

Fewer faster particles with $v_z > \frac{\omega}{k}$

More slower particles with $v_z < \frac{\omega}{k}$

$\frac{\omega}{k}$

v_z

⇒ Damping

ϕ

z

3.

Due to beam of fast particles ⇒

$F_0(v_z)$

Fewer with $v_z \lesssim \frac{\omega}{k}$

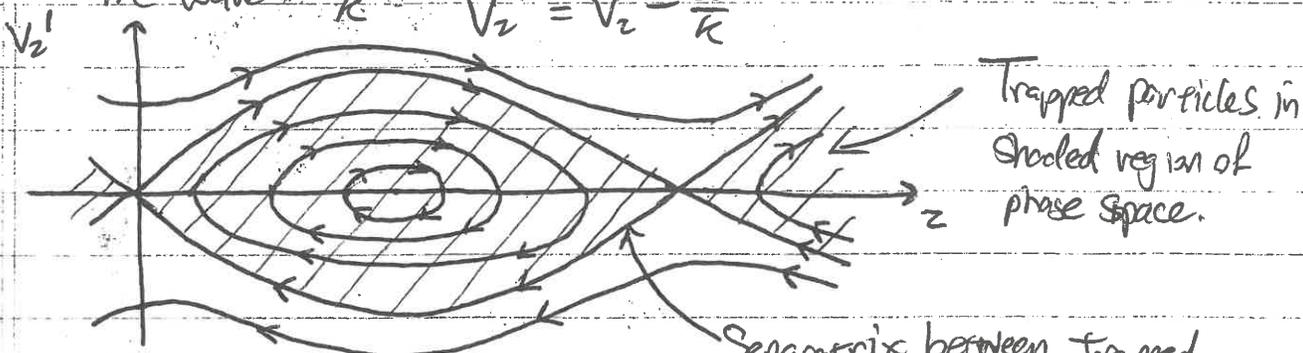More with $v_z \gtrsim \frac{\omega}{k}$

$\frac{\omega}{k}$

v_z

Bump-on-tail Instability:

Leads to unstable growth of E fluctuations from free energy in $F_0(v_z)$.
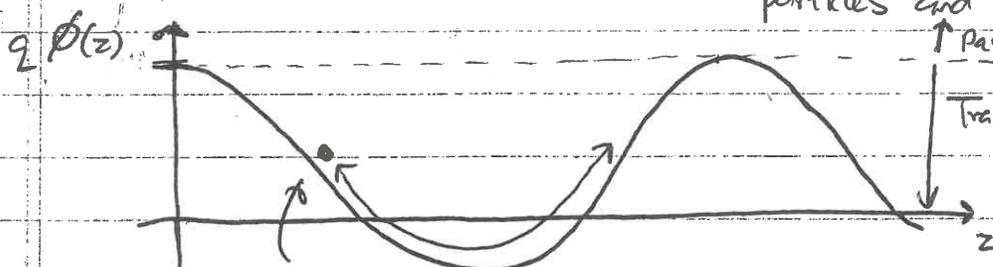
II. Phase Mixing Interpretation

A. Phase Space Pbe (V_z, z)

1. Transform to frame of reference moving at phase velocity of the wave $\frac{\omega}{k}$: $\quad V_z' = V_z - \frac{\omega}{k}$
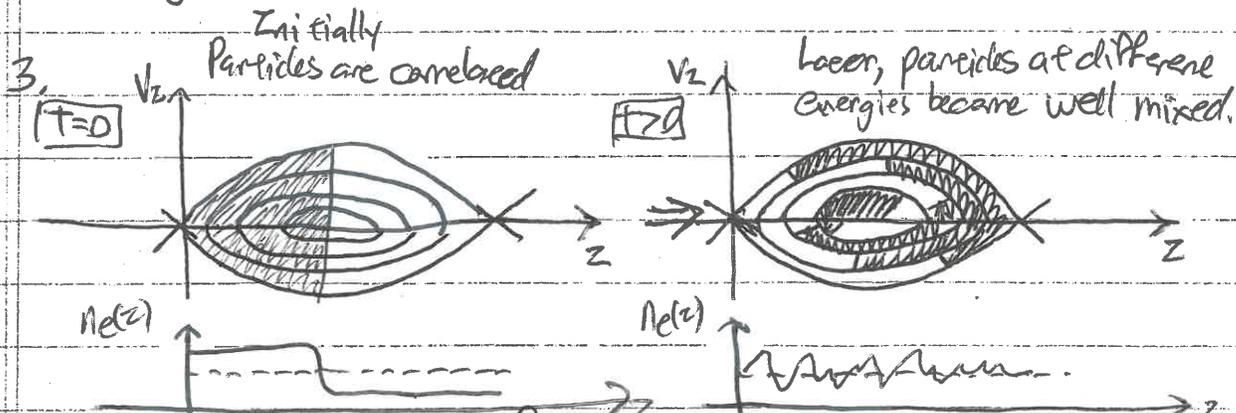

Trapped particles in shaded region of phase space.

Separatrix between trapped particles and passing particles

↑ Passing

Trapped

Particles trapped in potential well (many at phase speed $\frac{\omega}{k}$)

2. Bounce Frequency of Trapped particles decreases with increasing energy, reaching zero at separatrix.

3.

Initially Particles are correlated

$t=0$

Later, particles at different energies become well mixed.

$t > 0$

a. Since potential $-\nabla^2\phi = \frac{4\pi}{c}\int d^3v \, q_s f_s$ depends on integration over f_s, this phase mixing leads to an averaging out of the integral ⟹ damping of the wave.

III. Cold Beam Instabilities

A. Cold Beam Distribution:

1. A beam with number density n_j and velocity $\underline{v} = V_j \hat{\underline{z}}$ is given by

$$f_0(\underline{v}) = n_j \, \delta(v_x)\, \delta(v_y)\, \delta(v_z - V_j)$$

2. Since all particles have the same velocity, the beam has zero temperature $\Rightarrow$ no thermal spread of velocities
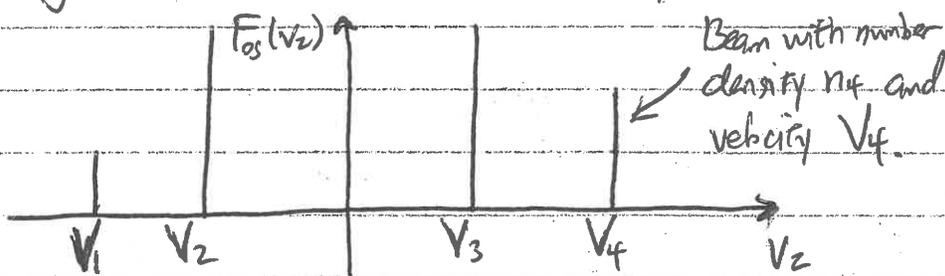
3. For a species s, we can have multiple beams,

$$f_{0s}(\underline{v}) = \sum_j n_{js} \, \delta(v_x)\, \delta(v_y)\, \delta(v_z - V_{js})$$

4. The one-dimensional distribution function is written

$$F_{0s}(v_z) = \frac{1}{n_{0s}} \int_{-\infty}^{\infty} f_{0s}(\underline{v})\, dv_x\, dv_y = \sum_j \frac{n_{js}}{n_{0s}} \, \delta(v_z - V_j)$$

where $n_0 = \sum_j n_j$ is the total number density of all beams.

5. Distribution:

Beam with number density n_4 and velocity V_4.

B. Dispersion Relation for Electrostatic Waves

1. The Laplace-Fourier Solution for electrostatic waves gives a Dispersion Relation

$$D(\underline{k}, p) = 1 - \sum_s \frac{\omega_{ps}^2}{k^2} \int_{-\infty}^{\infty} dv_z \, \frac{F_{0s}}{(v_z - \frac{ip}{k})^2} = 0$$

2. Using $\sum_j \omega_{ps}^2 \frac{n_{js}}{n_{0s}} = \sum_j \frac{n_{0s}\, q_s^2}{\epsilon_0\, m_s} \frac{n_{js}}{n_{0s}} = \sum_j \omega_{pj}^2$ where $\omega_{pj}^2 \equiv \frac{n_j\, q_s^2}{\epsilon_0\, m_s}$,

we obtain: $\displaystyle D(\underline{k}, p) = 1 - \sum_j \frac{\omega_{pj}^2}{k^2} \frac{1}{(V_{js} - \frac{ip}{k})^2} = 0$

II B. (Continued)

3. <u>NOTE</u>: Here we take a complex $\omega = ip$
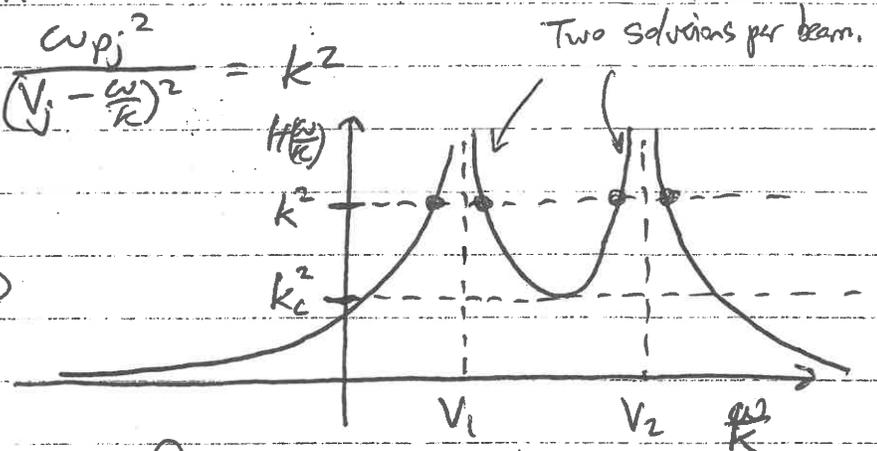   (we may write $\omega = \omega_r + i\gamma$ to choose real & imaginary parts).

So
$$D(k,\omega) = 1 - \sum_j \frac{\omega_{pj}^2}{k^2} \frac{1}{(V_j - \frac{\omega}{k})^2} = 0$$

4. Solutions occur when
$$H\left(\frac{\omega}{k}\right) = \sum_j \frac{\omega_{pj}^2}{(V_j - \frac{\omega}{k})^2} = k^2$$

a. We may plot this as

For two beams $\Rightarrow$

Two solutions per beam.

b. For $k > k_c$, there exise four real solutions $\Rightarrow \gamma = 0$.

c. But, for $k < k_c$, one obtains two real & two imaginary solutions.
   Since $H\left(\frac{\omega}{k}\right)$ is real, the roots come in complex conjugate pairs
   $$\begin{array}{l} \omega_+ = \omega_r + i\gamma \\ \omega_- = \omega_r - i\gamma \end{array}\Bigg\} \text{ Here we assume } \gamma > 0.$$

d. Thus, the root $\omega_+ = \omega_r + i\gamma$ gives a time dependence
   $$\sim e^{-i\omega_r t} \, e^{+\gamma t}$$

   Growth of this ω_+ solution. $\Rightarrow$ UNSTABLE

5. a. Single Beam leads to $\omega = \pm \omega_p - kV \Rightarrow$ Always stable
   b. When two (or more) beams are present, for sufficiently long
      wavelengths $k < k_c$, the plasma is always unstable.

$$\boxed{\text{Two Stream Instability}}$$

II. (Continued)

C. Two Stream Instability: Equal and opposite beams.

1. Take $n_1 = n_2 = n_0$ $V_1 = -V_2 = V_0$

2. Thus

$$D(\underline{k},\omega) = 1 - \frac{\omega_p^2}{(\omega - kV)^2} - \frac{\omega_p^2}{(\omega + kV)^2} = 0 \qquad \omega_p^2 = \frac{n_0 q_s^2}{\varepsilon_0 m_s}$$

3. a. This equation is quadratic in ω^2 and can be expressed as

$$\omega^4 - \omega^2 2(\omega_p^2 + k^2 V^2) - k^2 V^2 (2\omega_p^2 - k^2 V^2) = 0$$

b. Solution: $\boxed{\omega^2 = \underbrace{\omega_p^2 + k^2 V^2}_{\text{positive definite}} \pm \underbrace{\sqrt{\omega_p^4 + 4\omega_p^2 k^2 V^2}}_{\text{positive definite}}}$

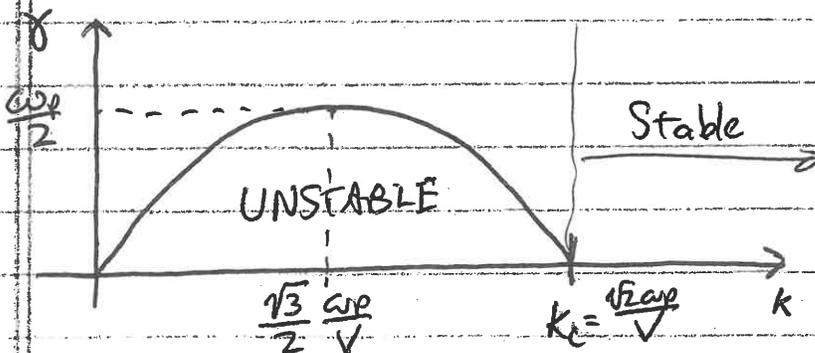4. a. For solution with negative sign, when

$$\omega_p^2 + k^2 V^2 < \left(\omega_p^4 + 4\omega_p^2 k^2 V^2\right)^{\frac{1}{2}},$$

then $\omega^2 < 0$ and the solution $\omega = \pm i\gamma$, leading to unstable growth.

b. Condition for this is $\boxed{k < k_c = \sqrt{2}\,\dfrac{\omega_p}{V}}$

c. The growth rate is

$$\gamma = \left[\left(\omega_p^4 + 4k^2 V^2 \omega_p^2\right)^{\frac{1}{2}} - \left(\omega_p^2 + k^2 V^2\right) \right]^{\frac{1}{2}}$$

Lecture #15 (Continued)

III. D. Weak Beam Approximation

1. Take a low density electron beam in a plasma at rest, $n_b \ll n_0$.

2. For $\omega_b^2 = \frac{n_b q^2}{\epsilon_0 m}$, we get $D(k,\omega) = 1 - \frac{\omega_p^2}{\omega^2} - \frac{\epsilon \omega_b^2}{(kV-\omega)^2} = 0$

3. Ordering: $\epsilon = \frac{\omega_b^2}{\omega_p^2} \ll 1$: Let $\omega = \omega_0 + \epsilon \omega_1$, and $\epsilon = \frac{n_b}{n_0} \ll 1$

   a. $\left[1 - \frac{\omega_p^2}{(\omega_0 + \epsilon \omega_1)^2} \right] (kV - (\omega_0 + \epsilon \omega_1))^2 = \epsilon \omega_b^2$

   b. Expanding: $\frac{\omega_p^2}{\omega_0^2 (1 + \epsilon \frac{\omega_1}{\omega_0})^2} \approx \frac{\omega_p^2}{\omega_0^2} \left(1 - \frac{2\epsilon \omega_1}{\omega_0} \right)$

4. $\mathcal{O}(1)$: $\left(1 - \frac{\omega_p^2}{\omega_0^2} \right) (kV - \omega_0)^2 = 0 \Rightarrow \omega_0 = \pm \omega_p \quad k = \frac{\pm \omega_p}{V}$

5. To solve at $\mathcal{O}(\epsilon)$, substitute solution ω_0 and k,

   a. $\left[1 - \frac{\omega_p^2}{\omega_p^2} \left(1 - \frac{2\epsilon \omega_1}{\omega_0} \right) \right] \left[kV - \omega_p - \omega_1 \right]^2 = \omega_b^2$

   $\Rightarrow \boxed{\omega_1^3 = -\frac{\omega_b^2 \omega_p}{2}}$

   b. Three roots are: $\omega_1 = \left(\frac{1}{2} \right)^{\frac{1}{3}} \omega_b^{\frac{2}{3}} \omega_p^{\frac{1}{3}}$ and $\omega_1 = \left(\frac{1}{2} \right)^{\frac{1}{3}} \omega_b^{\frac{2}{3}} \omega_p^{\frac{1}{3}} \left(-\frac{1}{2} \pm \frac{i\sqrt{3}}{2} \right)$

6. Root with positive sign gives unstable growth with

   a. $\omega_r = \omega_p \left[1 - \frac{1}{2} \left(\frac{1}{2} \right)^{\frac{1}{3}} \left(\frac{n_b}{n_0} \right)^{\frac{1}{3}} \right]$ and $\gamma = \frac{\sqrt{3}}{2} \left(\frac{1}{2} \right)^{\frac{1}{3}} \left(\frac{n_b}{n_0} \right)^{\frac{1}{3}} \omega_p$

   b. Growth rate $\gamma \propto n_b^{\frac{1}{3}}$

E. Buneman Instability

1. a. Background of cold ions at rest $n_e = n_i = n_0$

   b. Cold "Beam" of electrons moving through ions.

2. Dispersion Relation: $D(k,\omega) = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{(kV-\omega)^2} = 0$

3. Yields $\omega_r = \frac{1}{2} \left(\frac{m_e}{2m_i} \right)^{\frac{1}{3}} \omega_{pe}$ $\gamma = \frac{\sqrt{3}}{2} \left(\frac{m_e}{2m_i} \right)^{\frac{1}{3}} \omega_{pe}$

This fast-growing instability occurs when cold electrons stream through cold ions.