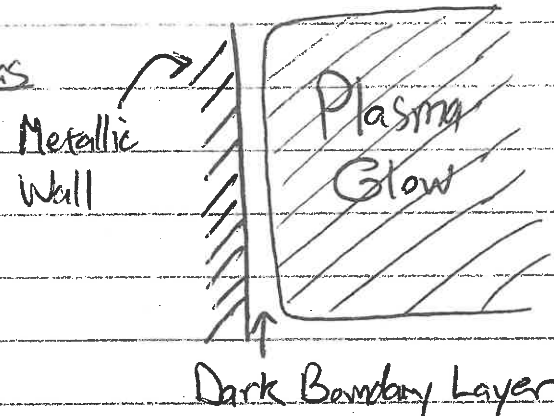
Lecture #19 Plasma Sheaths

I. Introduction to Plasma Sheaths

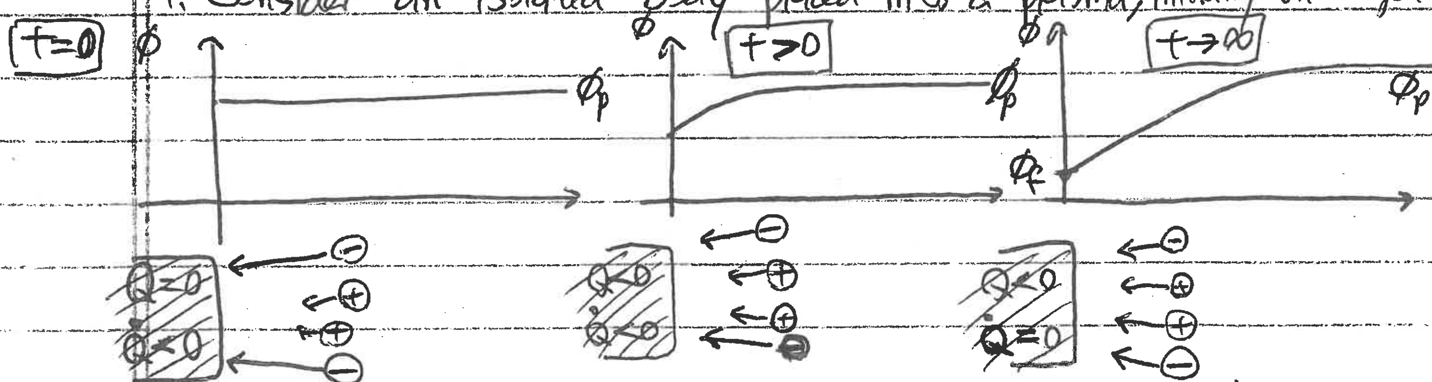
A. Bounded Plasmas



1. In this dark Boundary Layer, the plasma is depleted of the electrons needed to excite neutral atoms and produce the glow of the electric discharge
2. This dark region has a net positive charge, denoted the plasma sheath.

B. Floating Potential:

1. Consider an isolated body placed into a plasma, initially uncharged.



a) Since $v_{te} \gg v_{ti}$, electrons hit object more rapidly than ions ⇒ Object begins to charge negatively

b) Object gains negative potential, repelling electrons and attracting ions

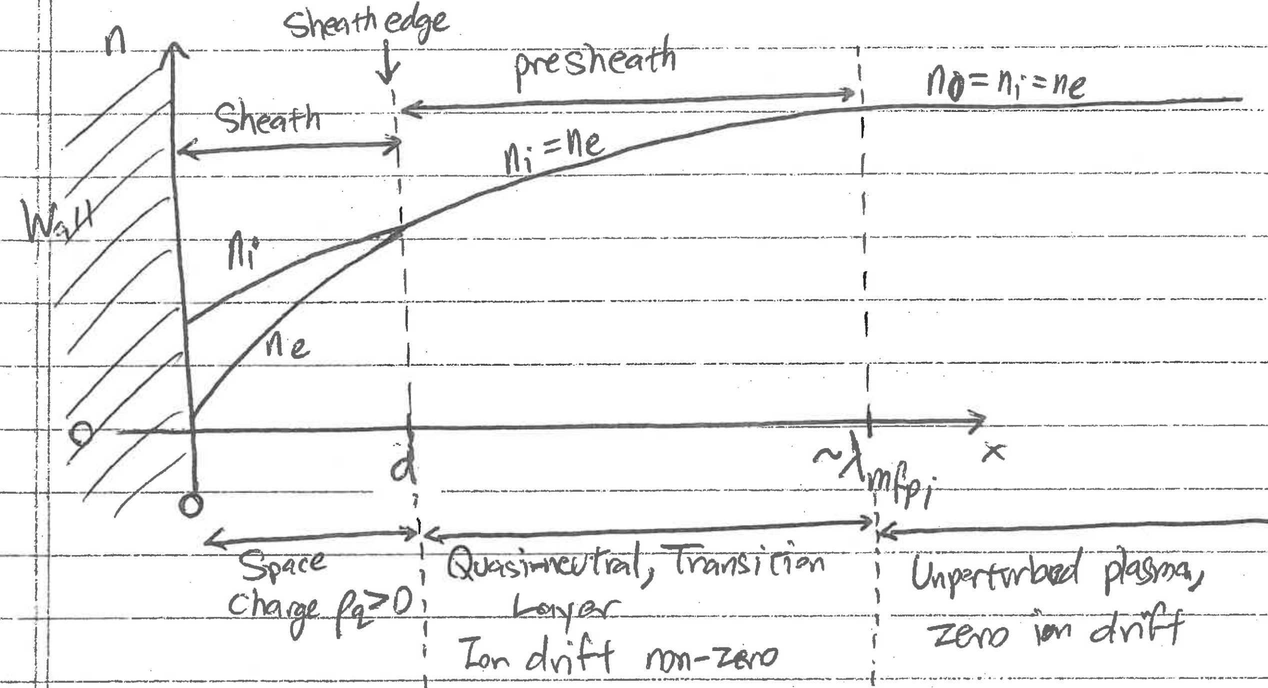
c) Net current is zero ⇒ Steady state!

2. Object charges negatively until attracted ions and repelled electrons lead to a net current of zero → This is the floating potential, ϕ_f

Lecture #20 (Continued) Howes (2)

I. (Continued)

C. The Structure of the Plasma Sheath:



II. The Child-Langmuir Law

A. Steady-State Solution for Sheath in Unmagnetized, Electrostatic Plasma

1. We can use Two-Fluid Theory to solve for the plasma sheath steady-state structure

2. Approximations: a. Hot, isothermal electrons
   b. Cold ions
   c. 1-D, Electrostatic system, $\underset{\sim}{E} = -\nabla\phi$

3. Two Fluid Equations for Electrostatic System

a. Continuity: $\dfrac{\partial n_s}{\partial t} + \nabla\cdot(n_s \underset{\sim}{U_s}) = 0$

b. Momentum: $m_s n_s \left[\dfrac{\partial \underset{\sim}{U_s}}{\partial t} + \underset{\sim}{U_s}\cdot\nabla\underset{\sim}{U_s}\right] = -\nabla p_s - q_s n_s \nabla\phi$

c. Poisson's Eq: $-\nabla^2\phi = \dfrac{\rho_q}{\epsilon_0}$

d. Equation of State: for Isothermal Elec: $p_e = n_e T_e$; Cold Ions: $T_i = 0$.

Lecture #20 (Continued)

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B. Electron density:

1. In steady state, $\frac{\partial}{\partial t} = 0$, so momentum equation is

$$\underbrace{m_e n_e \underset{\sim}{U_e} \cdot \nabla \underset{\sim}{U_e}}_{\text{Neglect electron inertia}} = -\nabla p_e - q_e n_e \nabla \phi$$

Balance electron pressure with electric field force

2. For 1-D system, with $p_e = n_e T_e$ and T_e = constant,

a. $$T_e \frac{\partial n_e}{\partial x} = e n_e \frac{\partial \phi}{\partial x}$$ where $q_e = -e$

b. $$\int_d^x \frac{1}{n_e}\frac{\partial n_e}{\partial x} dx = \int_d^x \frac{\partial}{\partial x}\left(\frac{e\phi}{T_e}\right) dx$$ Integrating from d (sheath edge) to x (within sheath)

c. $$\ln n_e \Big|_d^x = \frac{e}{T_e}(\phi)\Big|_d^x \Rightarrow \ln \frac{n_e(x)}{n_e(d)} = \frac{e}{T_e}\left[\phi(x) - \phi(d)\right]$$

3. Conditions at sheath edge: $\phi(d) \equiv 0$ ← set potential

$n_e(d) = n_i(d) \equiv n_d$

4. Thus, $$\boxed{n_e(x) = n_d e^{\frac{e\phi(x)}{T_e}}}$$ Boltzmann Distribution for Electrons

C. Ion Density

1. Steady State Ion Momentum Equation

a. $$m_i n_i \underset{\sim}{U_i} \cdot \nabla \underset{\sim}{U_i} = -\cancel{\nabla p_i}^{\,0} - q_i n_i \nabla \phi$$

Cold Ions $T_i = 0$.

b. Thus $$m_i \cancel{n_i} U_i \frac{\partial U_i}{\partial x} = -e \cancel{n_i} \frac{\partial \phi}{\partial x} \Rightarrow \frac{\partial}{\partial x}\left[\frac{1}{2} m_i U_i^2\right] = -\frac{\partial}{\partial x}\left[e\phi\right]$$

II C. (Continued)

2. Thus $\int_d^x \frac{\partial}{\partial x}\left[\frac{1}{2}m_i U_i^2 + e\phi\right]dx = 0$ Integrate from d to x.

a. $\frac{1}{2}m_i U_i^2(x) + e\phi(x) = \frac{1}{2}m_i U_i^2(d) + e\cancel{\phi(d)}^{\,0}$

b. At sheath edge, $U_i(d) \equiv u_0$

c. Solving for U_i:

$$U_i = \left[u_0^2 - \frac{2e\phi}{m_i}\right]^{\frac{1}{2}}$$

3. Steady State Continuity Eq. for Ions: $\nabla\cdot(n_i \underset{\sim}{U_i}) = 0$

a. $\int_d^x \frac{\partial}{\partial x}\left[n_i U_i\right]dx = 0$ Integrate from d to x

b. $n_i(x)U_i(x) = n_i(d)U_i(d) = n_d u_0$

c. $$\boxed{n_i(x) = n_d \frac{u_0}{U_i(x)} = n_d\left[1 - \frac{2e\phi}{m_i u_0^2}\right]^{-\frac{1}{2}}}$$

D. Poisson's Equation:

1. $\frac{\partial^2\phi}{\partial x^2} = -\frac{\rho_q}{\epsilon_0} = -\frac{1}{\epsilon_0}\left[n_i q_i + n_e q_e\right] = \frac{e n_d}{\epsilon_0}\left[\frac{n_e}{n_d} - \frac{n_i}{n_d}\right]$

2. $$\boxed{\frac{\partial^2\phi}{\partial x^2} = \frac{e n_d}{\epsilon_0}\left[e^{\frac{e\phi}{T_e}} - \left(1 - \frac{2e\phi}{m_i u_0^2}\right)^{-\frac{1}{2}}\right]}$$ Plasma Sheath Equation

Nonlinear equation for $\phi(x)$ within the sheath, $0 \le x \le d$.

Lecture #20 (Continued)

II. E. Solution of the Plasma Sheath Equation

1. a. We can solve the sheath equation analytically in the case that the wall is held at a sufficiently negative potential $\phi(0) = \phi_W$ such that $-\frac{e\phi_W}{T_e} >> 1$.

b. In this case, few electrons can overcome the potential barrier to reach the wall, and we may neglect the electron contribution to the space charge in the sheath.

c. We also assume $e\phi_W >> \frac{1}{2} m U_0^2$, that the energy gained by the ions is much larger than their initial kinetic energy.

2. In this limit, we obtain

a. $$\frac{\partial^2 \phi}{\partial x^2} = \frac{e n_d}{\epsilon_0}\left[\cancelto{0}{e^{\frac{e\phi}{T_e}}} - \left(\cancel{1} - \frac{2e\phi}{m_i U_0^2}\right)^{-\frac{1}{2}}\right]$$

b. $$\frac{\partial^2 \phi}{\partial x^2} = -\frac{e n_d}{\epsilon_0}\left(-\frac{2e\phi}{m_i U_0^2}\right)^{-\frac{1}{2}}$$

3. To clean up notation, we'll convert to dimensionless variables:

a. $\Phi \equiv \frac{-e\phi}{T_e}$, $M = \frac{U_0}{\left(\frac{T_e}{m_i}\right)^{\frac{1}{2}}} = \frac{U_0}{\underset{\uparrow}{C_s}}$, $\xi = \frac{x}{\lambda_D} = x\left(\frac{e^2 n_d}{\epsilon_0 T_e}\right)^{\frac{1}{2}}$

Ion acoustic speed

b. This yields $$\frac{\partial^2 \Phi}{\partial \xi^2} = \left(\frac{2\Phi}{M^2}\right)^{-\frac{1}{2}} = \frac{M}{\sqrt{2}}\Phi^{-\frac{1}{2}}$$

4. To solve this, we can multiply by $\frac{\partial \Phi}{\partial \xi}$ and integrate from $\frac{d}{\lambda_D}$ to ξ:

a. LHS: $$\int_{d/\lambda_D}^{\xi} \frac{\partial \Phi}{\partial \xi}\frac{\partial^2 \Phi}{\partial \xi^2} d\xi = \left(\frac{\partial \Phi}{\partial \xi}\right)^2\Bigg|_{d/\lambda_D}^{\xi} - \int_{d/\lambda_D}^{\xi} \frac{\partial \Phi}{\partial \xi}\frac{\partial^2 \Phi}{\partial \xi^2} d\xi$$

$u = \frac{\partial \Phi}{\partial \xi}$ $dv = \frac{\partial^2 \Phi}{\partial \xi^2} d\xi$

$du = \frac{\partial^2 \Phi}{\partial \xi^2} d\xi$ $v = \frac{\partial \Phi}{\partial \xi}$

II. E. (Continued)

b. Noting that $\Phi(\xi = \frac{d}{\lambda_D}) \overset{\phi(d)=0}{=} 0$, we obtain

$$\int_{d/\lambda_D}^{\xi} \frac{\partial \Phi}{\partial \xi}\frac{\partial^2 \Phi}{\partial \xi^2}\, d\xi = \frac{1}{2}\left[\left(\frac{\partial \Phi}{\partial \xi}\right)^2\Bigg|_{\xi} - \left(\frac{\partial \Phi}{\partial \xi}\right)^2\Bigg|_{d/\lambda_D}\right]$$

c. RHS: $\frac{M}{\sqrt{2}}\int_{d/\lambda_D}^{\xi} \Phi^{-\frac{1}{2}}\frac{d\Phi}{d\xi}\, d\xi = \frac{M}{\sqrt{2}}\Phi^{\frac{1}{2}}\Big|_{d/\lambda_D}^{\xi} + \frac{M}{\sqrt{2}}\frac{1}{2}\int_{d/\lambda_D}^{\xi} \Phi^{-\frac{1}{2}}\frac{d\Phi}{d\xi}\, d\xi$

$U = \Phi^{-\frac{1}{2}}$ $\quad dV = \frac{d\Phi}{d\xi}d\xi$

$dU = -\frac{1}{2}\Phi^{-\frac{3}{2}}\frac{d\Phi}{\partial \xi}$ $\quad V = \Phi$

d. Thus $\frac{M}{\sqrt{2}}\int_{d/\lambda_D}^{\xi} \Phi^{-\frac{1}{2}}\frac{d\Phi}{\partial \xi}\, d\xi = 2\frac{M}{\sqrt{2}}\Phi^{\frac{1}{2}}\Big|_{d/\lambda_D}^{\xi} = \sqrt{2}M\left[\Phi^{\frac{1}{2}} - \cancelto{0}{\Phi\left(\tfrac{d}{\lambda_D}\right)}\right]$

e. Therefore, we find

$$\frac{1}{2}\left[\left(\frac{\partial \Phi}{\partial \xi}\right)^2 - \cancelto{0}{\left(\frac{\partial \Phi}{\partial \xi}\right)^2\Bigg|_{d/\lambda_D}}\right] = \sqrt{2}M\Phi^{\frac{1}{2}}$$

f. Since the electric field $\frac{\partial \phi}{\partial x} \sim \frac{\partial \Phi}{\partial \xi}$ is small at the sheath edge $x=d$, we can neglect this contribution, leaving

$$\boxed{\frac{\partial \Phi}{\partial \xi} = 2^{\frac{3}{4}}M^{\frac{1}{2}}\Phi^{\frac{1}{4}}}$$

5. We can solve this equation by separation of variables:

a. $\int_{d/\lambda_D}^{\xi} \Phi^{-\frac{1}{4}}\frac{\partial \Phi}{\partial \xi}\, d\xi = \int_{d/\lambda_D}^{\xi} 2^{\frac{3}{4}}M^{\frac{1}{2}}\, d\xi$

II. E5, Continued)

b. $$\int_{d/\lambda_D}^{\xi} \Phi^{-\frac{1}{4}} \frac{d\Phi}{d\xi} d\xi = \Phi^{\frac{3}{4}} \Big|_{d/\lambda_D}^{\xi} + \frac{1}{4}\int_{d/\lambda_D}^{\xi} \Phi^{-\frac{1}{4}} \frac{d\Phi}{d\xi} d\xi$$

$U = \Phi^{-\frac{1}{4}}$ $\quad dv = \frac{d\Phi}{d\xi} d\xi$

$du = -\frac{1}{4}\Phi^{-\frac{5}{4}} \frac{d\Phi}{d\xi} d\xi \quad v = \Phi$

$$\Rightarrow \int_{d/\lambda_D}^{\xi} \Phi^{-\frac{1}{4}} \frac{d\Phi}{d\xi} d\xi = \frac{4}{3}\left[\Phi^{\frac{3}{4}} - \cancel{\Phi^{3/4}\Big|_{d/\lambda_D}}^{\,0}\right]$$

c. Thus $$\frac{4}{3}\Phi^{\frac{3}{4}} = 2^{\frac{3}{4}} M^{\frac{1}{2}}\left(\xi - \frac{d}{\lambda_D}\right)$$

d. Solving for the potential distribution within the sheath,

$$\boxed{\Phi = \left(\frac{3}{2}\right)^{4/3} 2\, M^{\frac{2}{3}} \left(\xi - \frac{d}{\lambda_D}\right)^{4/3}}$$

6. Consider the wall potential $\phi(0) = \phi_W \rightarrow \Phi_W$ at $\xi = 0$,

a. $$\Phi_W = \left(\frac{3}{2}\right)^{4/3} 2M^{2/3}\left(-\frac{d}{\lambda_D}\right)^{4/3}$$

b. We can convert $\Phi_W^{\frac{3}{2}}$ back to dimensional ~~units~~ variables, to obtain:

$$\phi_W^{\frac{3}{2}} = \frac{9}{4}\left(\frac{m_i}{2e}\right)^{\frac{1}{2}} \frac{n_d e U_o}{\epsilon_o} d^2$$

c. Noting that: Ion continuity equation gives $j_i = e n_i(x) U_i(x) = e n_d U_o$, constant!
we find

$$\phi_W^{3/2} = \frac{9}{4}\left(\frac{m_i}{2e}\right)^{\frac{1}{2}} \frac{j_i}{\epsilon_o} d^2$$

d. Solving for j_i: $$\boxed{j_i = \frac{4}{9}\epsilon_o \left(\frac{2e}{m_i}\right)^{\frac{1}{2}} \frac{\phi_W^{3/2}}{d^2}}$$ Child-Langmuir Law,

Lecture #20 (Continued)

II. E. (Continued)

7. a. The Child-Langmuir Law relates the ion current, j_i, in the sheath to the potential drop from the sheath edge, ϕ_w, and the sheath width, d.

b. This was originally formulated for the space-charge limited electron flow in a vacuum diode.

III. The Bohm Criterion

A. Stability of Plasma Sheath.

1. If the sheath dramatically violates quasi-neutrality, why doesn't it cause the charge perturbation to propagate into the plasma as an ion acoustic wave?

2. We can answer this question by considering the region near the sheath edge, $x=d$.

B. Behavior near Sheath Edge

1. In terms of the dimensionless variables, the full plasma sheath equation is

$$\frac{\partial^2 \Phi}{\partial \xi^2} = \left(1+\frac{2\Phi}{M^2}\right)^{-\frac{1}{2}} - e^{-\Phi}$$

2. Near $\xi = \frac{d}{\lambda_D}$ (sheath edge) $|\Phi| \ll 1$, so we can expand the RHS:

a. First, we multiply by $\frac{\partial \Phi}{\partial \xi}$ and integrate $\int_{d/\lambda_D}^{\xi} d\xi$ to give

$$\frac{1}{2}\left(\frac{\partial \Phi}{\partial \xi}\right)^2 = M^2\left[\left(1+\frac{2\Phi}{M^2}\right)^{\frac{1}{2}} - 1\right] + \left(e^{-\Phi} - 1\right)$$

b. Expand RHS for $\Phi \ll 1$

i) $\left(1+\frac{2\Phi}{M^2}\right)^{\frac{1}{2}} - 1 \simeq \left(\cancel{1} + \frac{\Phi}{M^2} - \frac{4\Phi^2}{8M^4} + \ldots\right) - \cancel{1}$

III. B.2 (Continued)

b. ii) $e^{-\Phi} + 1 = (\cancel{1} - \Phi + \frac{\Phi^2}{2} - \dots) - \cancel{1} = -\Phi + \frac{\Phi^2}{2}$

iii) Thus

$$\frac{1}{2}\left(\frac{\partial \Phi}{\partial \xi}\right)^2 = \cancel{\Phi} - \frac{\Phi^2}{2M^2} + \left(-\cancel{\Phi} + \frac{\Phi^2}{2}\right) = \frac{1}{2}\Phi^2\left(1 - \frac{1}{M^2}\right)$$

3. So, we find

$$\boxed{\frac{\partial \Phi}{\partial \xi} = \sqrt{1 - \frac{1}{M^2}}\,\Phi}$$

4. a) If $1 - \frac{1}{M^2} < 0$, solutions are oscillatory $\Rightarrow M^2 < 1$

b) If $1 - \frac{1}{M^2} > 0$, solutions are monotonic $\Rightarrow M^2 > 1$.

5a. We want monotonic solutions since we expect the sheath potential to decrease from 1) $\phi > 0$ unperturbed plasma
to 2) $\phi(x=d) = 0$ sheath edge
to 3) $\phi_w < 0$ wall

b. Thus, we require $\boxed{M \geq 1, \text{ or } U_0 \geq c_s.}$ → Bohm Criterion

$\Rightarrow$ $\boxed{\text{Ion flow toward the wall at sheath edge is supersonic}}$

c. Thus, no wave information can propagate from the wall into the plasma.

d. This supersonic flow is required for stability of the sheath.

Lecture #20 (Continued)

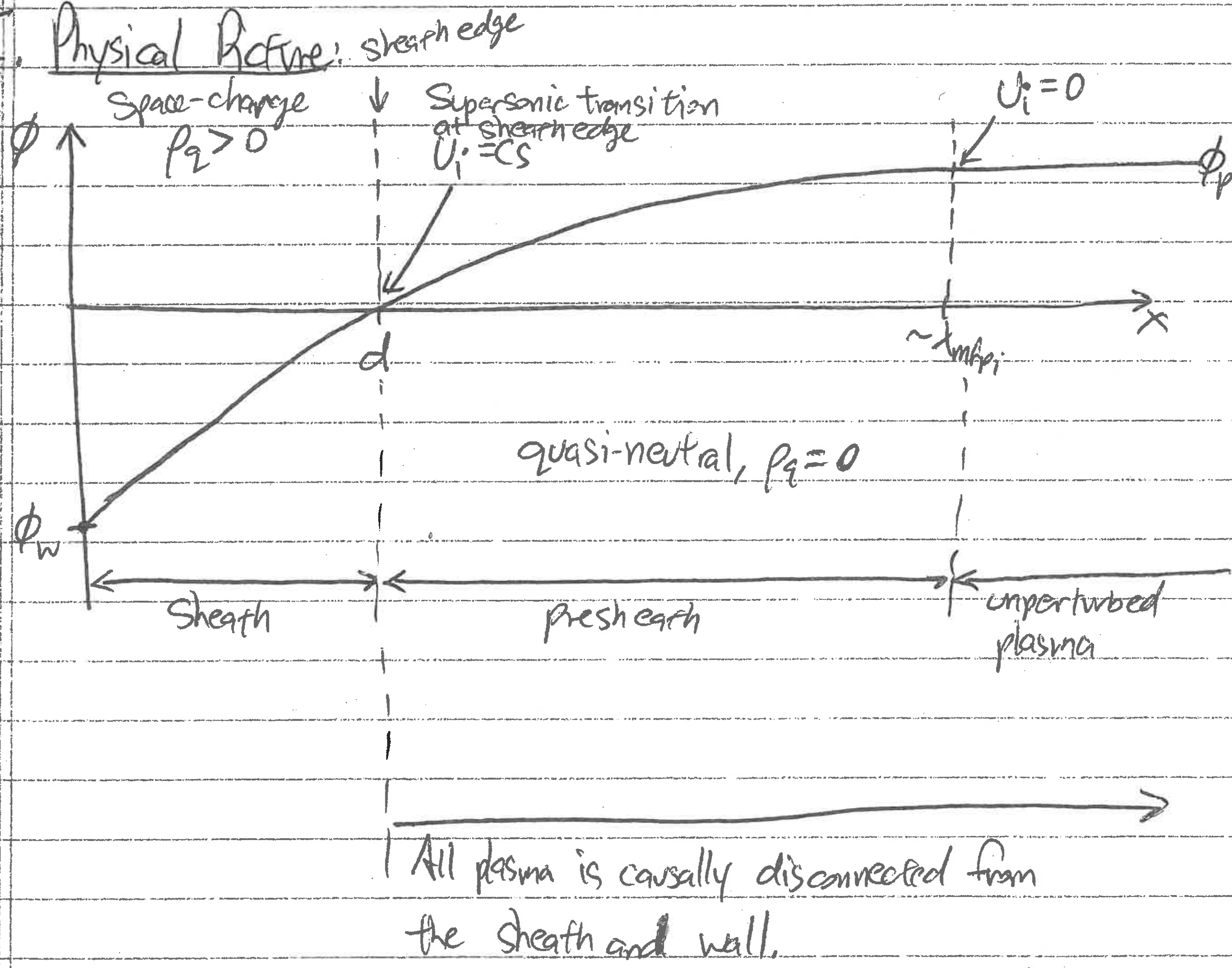
III.C. Presheath Stability:

1. One can determine a similar stability condition in the presheath where it is quasineutral $n_i = n_e = n_d e^{+\frac{e\phi}{T_e}}$.

2. The condition for stability requires $M \leq 1$ within the presheath.
   a. Velocities within the presheath are subsonic, $U_0 \leq C_s$.

3. Therefore, the only solution that satisfies both conditions is $U_0 = C_s \rightarrow$ Supersonic transition occurs at sheath edge.

IV. Physical Picture:



1. In a real plasma, sheath width d adjusts until it meets the condition $U_0 = C_s$.
2. This is typical for flow solutions that are trans-sonic (like solar wind)