Lecture #2: Parallel Propagating Waves & Preliminary Perpendicular Propagating Waves

I. Parallel Propagating Waves ($\underset{\sim}{k} \parallel \underset{\sim}{B}$)

A. For $\underset{\sim}{k} \parallel \underset{\sim}{B}$, the angle $\theta = 0$.

1. Solutions are easily found using the "Tangent" form of the dispersion relation

$$\tan^2\theta = \frac{-P\,(n^2-R)(n^2-L)}{(Sn^2-RL)(n^2-P)}$$

2. For $\theta=0$, $\tan\theta = 0$, so we must have

$$\left.\begin{array}{l} \text{a. } P=0 \\ \text{b. } n^2=R \\ \text{c. } n^2=L \end{array}\right\} \text{Three roots for parallel propagation}$$

3. We can understand more about these solutions if we look at the electric field eigenfunction using matrix form of dispersion relation

a. For $\theta=0$, we have

$$\begin{pmatrix} S-n^2 & -iD & 0 \\ iD & S-n^2 & 0 \\ 0 & 0 & P \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

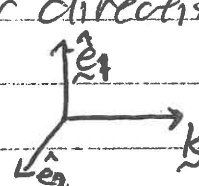
b. Recall that we have performed a Fourier transform, so (E_x, E_y, E_z) are really the Fourier coefficients $\underset{\sim}{E}(\underset{\sim}{k})$ such that

$$\underset{\sim}{E}_1(\underset{\sim}{x},t) = \underset{\sim}{E}_1(\underset{\sim}{k})\, e^{i[\underset{\sim}{k}\cdot\underset{\sim}{x} - \omega(\underset{\sim}{k})t]}$$

B. Longitudinal Wave Plasma Oscillation: ($P=0$)

1. Recall (from PHYS:4731 Lect #22) that longitudinal and transverse are defined with respect to the wave vector direction $\hat{\underset{\sim}{k}}$.

$$\underset{\sim}{E}_1 = E_{T_1}\,\hat{\underset{\sim}{e}}_1 + E_{T_2}\,\hat{\underset{\sim}{e}}_2 + E_L\,\hat{\underset{\sim}{k}}$$



$$\hat{\underset{\sim}{e}}_1 \times \hat{\underset{\sim}{e}}_2 = \hat{\underset{\sim}{k}}$$

2. In a magnetized plasma, the direction of the magnetic field $\hat{\underset{\sim}{b}} = \frac{\underset{\sim}{B}}{|\underset{\sim}{B}|}$ is another important direction.

a. For the case of parallel wavevectors ($\underset{\sim}{k} \parallel \hat{\underset{\sim}{b}}$), we have $\hat{\underset{\sim}{k}} = \hat{\underset{\sim}{b}} = \hat{\underset{\sim}{z}}$, $\hat{\underset{\sim}{e}}_1 = \hat{\underset{\sim}{x}}$, $\hat{\underset{\sim}{e}}_2 = \hat{\underset{\sim}{y}}$

Lecture #2: (Continued)

I. B. (Continued)

3. The eigenfunction for the $P=0$ root has $E_z \neq 0$, $E_x = E_y = 0$.

$\Rightarrow \underset{\sim}{E}_1 = E_1 \hat{\underset{\sim}{k}}$

a. This is a longitudinal mode since $\hat{\underset{\sim}{k}} \times \underset{\sim}{E}_1 = 0$.

4. Solving for Frequency: $P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} = 0$

$\Rightarrow \boxed{\omega^2 = \omega_{pi}^2 + \omega_{pe}^2}$ (remember we assume ion & electron plasma)

a. This is just the usual dispersion relation for the plasma oscillation.

b. This longitudinal mode has motion along the magnetic field.

$\Rightarrow$ Lorentz force $\sim \underset{\sim}{v} \times \underset{\sim}{B} = 0$

c. This is just the usual plasma oscillation in the direction along the mean field!

C. Right-hand Polarized Transverse Wave $(n^2 = R)$

1. NOTE: $S - n^2 = S - R = \frac{1}{2}(R+L) - R = -\frac{1}{2}(R-L) = -D$.

Thus, we get
$$\begin{pmatrix} -D & -iD & 0 \\ iD & -D & 0 \\ 0 & 0 & P \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

a. Eigenfunction: Take $E_x = E_0$, then $-D(E_0) - iDE_y = 0$

$\Rightarrow E_y = iE_0$

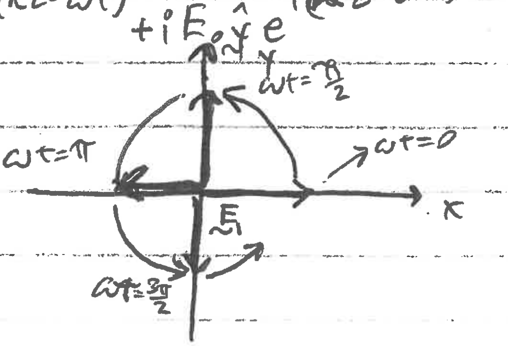
b. For $n^2 = R$, eigenfunction is $\underset{\sim}{E}_1 = (E_0, iE_0, 0)$

2. Right-handed polarization: $\underset{\sim}{E}(\underset{\sim}{x}, t) = E_0 \hat{\underset{\sim}{x}} e^{i(kz - \omega t)} + iE_0 \hat{\underset{\sim}{y}} e^{i(kz - \omega t)}$

a. The real component at $z=0$ becomes

Circular polarization $\rightarrow$ $E(z=0,t) = E_0 \cos\omega t \, \hat{\underset{\sim}{x}} + E_0 \sin\omega t \, \hat{\underset{\sim}{y}}$

b. This rotates in a counter-clockwise (right-hand) sense with time.

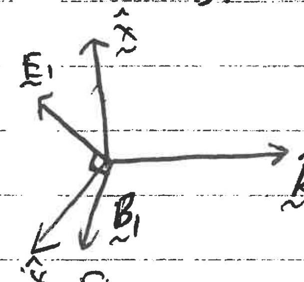


I.C. (Continued)

3. Transverse, Electromagnetic Waves

a. Since $\underset{\sim}{E}_1 = E_0\hat{x} + iE_0\hat{y}$, $\underset{\sim}{E}_1 \times \underset{\sim}{\hat{k}} \neq 0 \Rightarrow$ Waves are transverse.

b. Faraday's Law gives $\underset{\sim}{B}_1 = \frac{\underset{\sim}{k}}{\omega} \times \underset{\sim}{E}_1$, so



Electromagnetic

$$\underset{\sim}{B}_1 = \frac{k}{\omega}\ \underset{\sim}{\hat{k}} \times (E_0\hat{x} + iE_0\hat{y}) = \frac{kE_0}{\omega}\hat{y} - \frac{ikE_0}{\omega}\hat{x}$$

$$\Rightarrow \underset{\sim}{E}_1 = E_0(\hat{x} + i\hat{y}), \quad \underset{\sim}{B}_1 = \frac{-ikE_0}{\omega}(\hat{x} + i\hat{y})$$

4. Solution for Frequency: $n^2 = R$

a. $$\frac{c^2k^2}{\omega^2} = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + \omega_{cs})} = 1 - \frac{\omega_{pi}^2}{\omega(\omega+\omega_{ci})} - \frac{\omega_{pe}^2}{\omega(\omega+\omega_{ce})}$$

where $\omega_{ps}^2 = \frac{n_s q_s^2}{\epsilon_0 m_s}$ and $\omega_p^2 = \omega_{pi}^2 + \omega_{pe}^2 \simeq \omega_{pe}^2$

and $\omega_{cs} = \frac{q_s B_0}{m_s}$ (and this carries sign of charge q_s).

b. This solution may be rewritten:

$$\frac{k^2c^2}{\omega^2} = \frac{(\omega+\omega_L)(\omega-\omega_R)}{(\omega+\omega_{ce})(\omega+\omega_{ci})}$$

where $$\omega_L \equiv \frac{\omega_{ce}+\omega_{ci}}{2} + \frac{1}{2}\sqrt{(\omega_{ce}+\omega_{ci})^2 + 4(\omega_p^2 - \omega_{ce}\omega_{ci})}$$

$$\omega_R = -\frac{(\omega_{ce}+\omega_{ci})}{2} + \frac{1}{2}\sqrt{(\omega_{ce}+\omega_{ci})^2 + 4(\omega_p^2 - \omega_{ce}\omega_{ci})} = \omega_L - (\omega_{ce}+\omega_{ci}$$

NOTE: $\omega_L\omega_R = \omega_p^2 - \omega_{ce}\omega_{ci}$

D. Left-hand Polarized Transverse Waves ($n^2 = L$)

1. Eigenfunction $\underset{\sim}{E}_1 = (E_0, -iE_0, 0) \Rightarrow$ Left-hand circular polarization
2. Similar to the R solution but with

$$\frac{k^2c^2}{\omega^2} = \frac{(\omega-\omega_L)(\omega+\omega_R)}{(\omega-\omega_{ce})(\omega-\omega_{ci})}$$

I. (Continued)

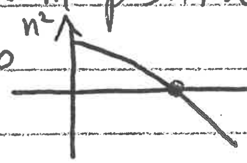
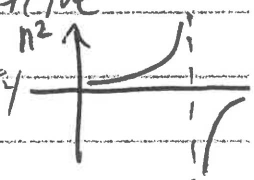
E. Limits of R & L waves:

1. **NOTE**: For most plasmas, $\omega_{ce} < \omega_{pe}$. We'll consider only this case.
   (This corresponds to $\frac{V_A^2}{c^2} < \frac{m_e}{m_i}$)

2. Resonances and cutoffs:

   a. Index of refraction $n = \frac{ck}{\omega}$. $n^2 = \frac{(\omega \pm \omega_L)(\omega \mp \omega_R)}{(\omega \pm \omega_{ce})(\omega \pm \omega_{ci})}$ Sign $\binom{R}{L}$

   b. For $n^2 > 0$, we have a propagating wave
   $n^2 < 0$, we have an evanescent (damped) solution.

   c. Values of n^2 can switch from positive to negative either by going through zero  or infinity 

   d. $\boxed{n^2 \to \infty \Rightarrow \text{Resonance}}$

   e. $\boxed{n^2 \to 0 \Rightarrow \text{Cutoff}}$

3. For R mode, let's simplify using $|\omega_{ce}| \gg \omega_{ci}$ and $\omega_{pe}^2 \gg \omega_{pi}^2$

   a. $\Rightarrow \omega_{ce} + \omega_{ci} \simeq \omega_{ce}$ and $\omega_{pe}^2 + \omega_{pi}^2 \simeq \omega_{pe}^2$

   b. Also, let's take absolute values in the formula, $\omega_{ce} = -|\omega_{ce}|$

4. R mode becomes:

$$n^2 = \frac{(\omega + \omega_L)(\omega - \omega_R)}{(\omega - |\omega_{ce}|)(\omega + \omega_{ci})}$$

where $\omega_R \simeq \frac{|\omega_{ce}|}{2} + \frac{1}{2}\sqrt{\omega_{ce}^2 + 4(\omega_p^2 + |\omega_{ce}|\omega_{ci})}$

and $\omega_L \simeq \omega_R - |\omega_{ce}|$. Thus $\omega_R > \omega_L$.

a. $n^2 > 0$ when $\omega > \omega_R$ or $\omega < |\omega_{ce}|$
$n^2 < 0$ (evanescent) when $|\omega_{ce}| < \omega < \omega_R$

Lecture #2 (Continued) Howes ⑤

I. E. 4. (Continued)

b. Resonance ($n^2 \to \infty$) occurs when $\omega = |\omega_{ce}|$.

c. Cutoff ($n^2 \to 0$) occurs when $\omega = \omega_R$.

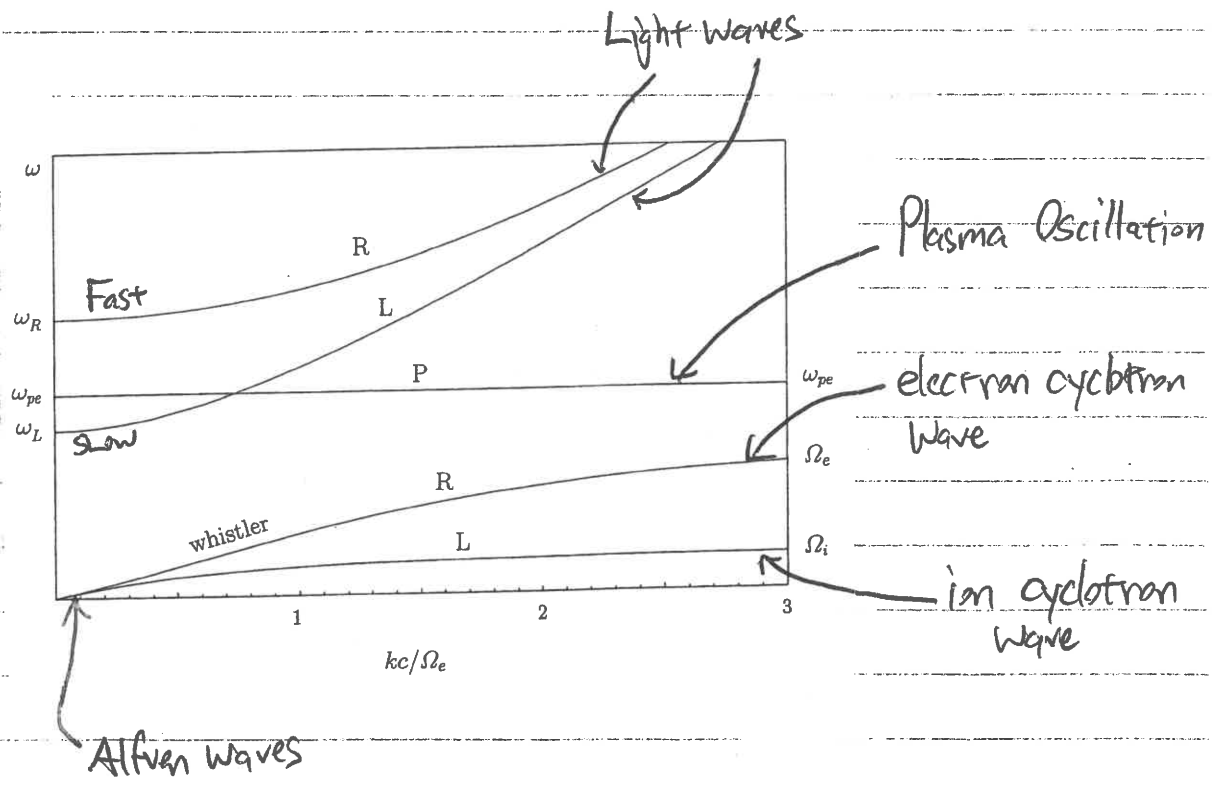
5. For the L mode, a. $n^2 > 0$ when $\omega > \omega_L$ $\omega < \omega_{ci}$

$n^2 < 0$ when $\omega_{ci} < \omega < \omega_L$

b. Resonance ($n^2 \to \infty$) when $\omega = \omega_{ci}$.

Cutoff ($n^2 \to 0$) when $\omega = \omega_L$.

6. $\omega(k)$ for modes with parallel wave vector: $\underset{\sim}{k} = k\hat{\underset{\sim}{z}}$.



7. Frequency Limits:

a. Alfven Waves: Low frequency $\omega \ll \omega_{ci} \ll |\omega_{ce}| < \omega_{pe}$

$$\frac{k^2c^2}{\omega^2} = \frac{(\cancel{\omega}+\omega_L)(\cancel{\omega}-\omega_R)}{(\cancel{\omega}-|\omega_{ce}|)(\cancel{\omega}+\omega_{ci})} \simeq \frac{+\omega_L\omega_R}{+|\omega_{ce}|\omega_{ci}} = \frac{\omega_{pe}^2 \cancel{+|\omega_{ce}|\omega_{ci}}}{|\omega_{ce}|\omega_{ci}} = \frac{\frac{n_0 q_e^2}{\epsilon_0 m_e}}{\frac{|q_e|B_0}{m_e}\frac{q_i B_0}{m_i}}$$

$$\Rightarrow \frac{k^2}{\omega^2}\frac{1}{\cancel{\mu_0\epsilon_0}} = \frac{n_0 m_i}{\cancel{\epsilon_0} B_0^2} \Rightarrow \boxed{\omega^2 = k^2 V_A^2} \text{ (Same for both L \& R modes)}$$

Lecture #2 (Continued)

I. E.7. (Continued)

b. Ion Cyclotron Waves: $\omega \simeq \omega_{ci}$ (L mode)

$$\Rightarrow (\omega - \omega_{ci}) \simeq \frac{-\omega_{ci}^3}{k^2 V_A^2}$$

c. Whistler Waves: $\omega_{ci} \ll \omega \ll |\omega_{ce}|$ (R mode)

$$\Rightarrow \omega = \frac{k^2 c^2}{\omega_{pe}^2} |\omega_{ce}|$$

d. Electron Cyclotron Waves: $\omega \simeq |\omega_{ce}|$ (R mode)

$$\Rightarrow (\omega - |\omega_{ce}|) \simeq \frac{\omega_{pe}^2}{|\omega_{ce}|} \left(1 - \frac{k^2 c^2}{\omega^2}\right)^{-1}$$

e. Left- and Right-hand polarized Light Waves (R & L modes)

1. $\omega > \omega_R$ or $\omega > \omega_L$. $\gg |\omega_{ce}| \gg \omega_{ci}$

2. For R mode:

$$\frac{k^2 c^2}{\omega^2} = \frac{(\omega + \omega_L)(\omega - \omega_R)}{(\omega - \cancel{|\omega_{ce}|})(\omega + \cancel{\omega_{ci}})} \simeq \frac{\omega^2 + \omega(\omega_L - \omega_R) - \omega_L \omega_R}{\cancel{\omega^2}}$$

$$\Rightarrow k^2 c^2 = \omega^2 - \cancel{\omega |\omega_{ce}|} - (\omega_{pe}^2 + \cancel{|\omega_{ce}| \omega_{ci}}) \Rightarrow \boxed{\omega^2 \simeq c^2 k^2 + \omega_{pe}^2}$$

Usual modified lightwave dispersion relation.

II. Perpendicular propagating waves ($\underline{k} \perp \underline{B}$)

A. 1. For $\underline{k} \perp \underline{B}$, we have $\theta = \frac{\pi}{2}$ or $\tan\theta \to \infty$

2. Solutions occurs for

a. $n^2 = P$

b. $n^2 = \frac{RL}{S}$

$\Bigg\}$ Two roots for perpendicular propagation.

II. A. (Continued)

3. For $\theta = \frac{\pi}{2}$, Eigenfunctions are given by
$$\begin{pmatrix} S & -iD & 0 \\ iD & S-n^2 & 0 \\ 0 & 0 & P-n^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

B. Ordinary Mode ($n^2 = P$)

1. For the solution $n^2 = P$, the eigenfunction is $\underset{\sim}{E}_1 = (0, 0, E_0)$
2. Transverse Wave:
   a. Now we have $\underset{\sim}{k} = k\hat{\underset{\sim}{x}}$, so $\underset{\sim}{k} \times \underset{\sim}{E}_1 \neq 0 \Rightarrow$ Transverse
   b. Likewise, this is an electromagnetic wave.

3. Solution for frequency: $n^2 = P$

   a. $\frac{k^2c^2}{\omega^2} = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \Rightarrow k^2c^2 = \omega^2 - (\omega_{pi}^2 + \omega_{pe}^2)$

   $\Rightarrow \boxed{\omega^2 = k^2c^2 + \omega_p^2}$ The usual Modified light wave.

   b. Motions due to the electric field are parallel to the magnetic field, so there is no $\underset{\sim}{v} \times \underset{\sim}{B}$ Lorentz Force.
   $\Rightarrow$ The magnetic field has no effect on waves of this polarization (parallel to $\underset{\sim}{B}$).

   $\Rightarrow$ $\boxed{\text{Ordinary Mode}}$ (linearly polarized).