Lecture #23: Buckingham Pi Theorem & Nondimensionalization

I. Simple Explanation of Buckingham Pi Theorem

A. Dimensional Homogeneity

1. Any physical relation can be expressed as a sum of terms, where all summed terms must have the same units,

$$\Psi_1 + \Psi_2 + \Psi_3 + \ldots + \Psi_N = 0$$

2. We can make any such equation dimensionless by dividing through by any one of the Ψ_j,

$$1 + \frac{\Psi_2}{\Psi_1} + \frac{\Psi_3}{\Psi_1} + \ldots + \frac{\Psi_N}{\Psi_1} = 0$$

3. In this dimensionless equation, one can change units at will without changing the equation ($m \rightarrow cm$ or in. or light years)

a. One can express the equation above as a function of n physical quantities v_i,

$$f(v_i) = 0$$

b. This expression can also include — beyond variables like position, velocity, acceleration, etc. — physical constants or other parameters, including the actual physical quantity being sought (oscillation frequency, pressure drop along a pipe, etc.)

B. The Buckingham Approach

1. Buckingham suggestion the expression $f(v_i)=0$ can be manipulated into the following form,

$$g(\Pi_1, \Pi_2, \ldots, \Pi_{n-k}) = 0$$

where the n physical quantities can be expressed in terms of k distinct fundamental units (length, mass, time, charge, etc.), where each of the <u>dimensionless quantities</u> Π_j can be written

$$\Pi_j = (v_1)^{\alpha_{j1}} (v_2)^{\alpha_{j2}} \ldots (v_n)^{\alpha_{jn}}$$

where the α_{ji} are rational powers.

2. In general, we can pull out one of the dimensionless Π_j to obtain a new relation

$$\Pi_1 = h(\Pi_2, \Pi_3, \ldots, \Pi_{n-k}, \nu)$$

a. The index ν admits that there may be more than one root of Π_1 that solves the equation above.

3. We'll see the power of this very general dimensional analysis approach to provide important constraints on the solutions to many physical problems.

I. B. (Continued)

4. Note that it is still critical to apply physical intuition to gain insight into the physical behavior of solutions.

   a. Often we seek the scaling of the solution in terms of the fundamental parameters

5. Note that if $n-k=1$ dimensionless parameter, you can write

$$f(\Pi_1)=0=(\Pi_1)^\alpha - K = 0$$

constant

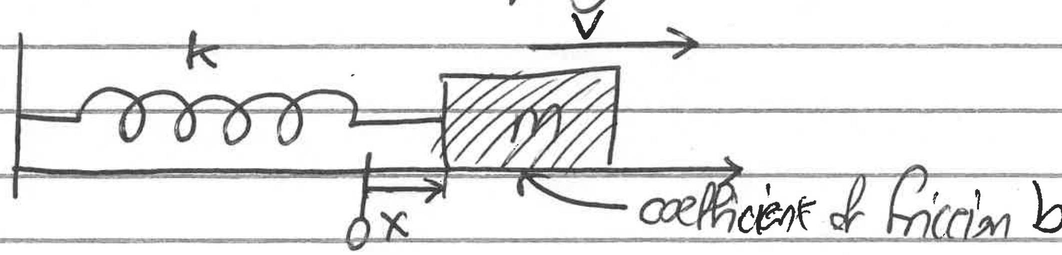
   a. Thus, the solution can be expressed as

$$\boxed{\Pi_1 = K^{\frac{1}{\alpha}}}$$

II. Examples

A. Damped, Simple Harmonic Oscillator

1. Consider a mass on a spring with friction



2. The dynamical equation for the forces in this system,

$$\boxed{m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=0}$$

II. A. Continued

3. What are the fundamental dimensionless parameters upon which the frequency of the system depends?

4. Fourier transform Solution

a. Let us first solve this problem using a Fourier transform approach

(We'll ignore subtleties of convergence of the transform here that require a Laplace transform approach in the case of an imaginary component to the frequency)

b. FT $\rightarrow$ $\frac{\partial}{\partial t} \rightarrow -i\omega$

i) $-\omega^2 m \cancel{\hat{x}} - i\omega b \cancel{\hat{x}} + k\cancel{\hat{x}} = 0$

ii) $\omega^2 + i\frac{b}{m}\omega - \frac{k}{m} = 0$

iii) Quadratic equation solution: $\omega = \dfrac{-i\frac{b}{m} \pm \sqrt{-\frac{b^2}{m^2} + 4\frac{k}{m}}}{2}$

$$\boxed{\omega = -i\frac{b}{2m} \pm \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$$

iv) Thus: a. Real frequency: $\omega_r = \pm\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$

b. Damping rate: $\gamma = -\frac{b}{2m}$

5. Express frequency in dimensionless units:

a. $\omega = -i\frac{b}{2m} \pm \left(\frac{k}{m}\right)^{\frac{1}{2}}\left(1 - \frac{b^2}{4km}\right)^{\frac{1}{2}}$ ← divide by $\left(\frac{k}{m}\right)^{\frac{1}{2}}$

b. $\omega\left(\frac{m}{k}\right)^{\frac{1}{2}} = \pm\left(1 - \frac{b^2}{4km}\right)^{\frac{1}{2}} - i\left(\frac{b^2}{4km}\right)^{\frac{1}{2}}$

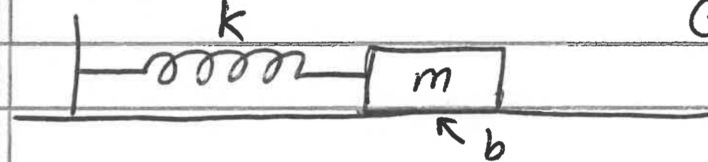
II.A. (Continued)

5. (Continued)

c. Define
$$\boxed{\begin{aligned}\bar{\omega} &= \omega\left(\frac{m}{k}\right)^{\frac{1}{2}} \\ \bar{b} &= \left(\frac{b^2}{4km}\right)^{\frac{1}{2}}\end{aligned}}$$
Dimensionless

d. Thus, the dispersion relation for the frequency becomes
$$\boxed{\bar{\omega} = \pm\left(1-\bar{b}^2\right)^{\frac{1}{2}} - i\bar{b}}$$

6. Pi Theorem:



a. Physical Quantities

$$\left\{\begin{array}{ll} m & [M] \\ b & \left[\frac{M}{T}\right] \\ k & [M/T^2] \end{array}\right.$$
$$\{\omega \quad [1/T]$$

b. m, b, k are physical parameters (constants) of the system
ω is a dynamical variable

c. $n = 4$ physical quantities
$k = 2$ (M, T) fundamental units

$m = n - k = 2$ dimensionless quantities

d. Create dimensionless quantities

$$\pi_1 = \frac{m\omega^2}{k} \qquad \left[\cancel{M}\left(\frac{\cancel{T^2}}{\cancel{M}}\right)\frac{1}{\cancel{T^2}}\right]$$

$$\pi_2 = \frac{b\omega}{k} \qquad \left[\left(\frac{\cancel{M}}{\cancel{T}}\right)\left(\frac{\cancel{T^2}}{\cancel{M}}\right)\frac{1}{\cancel{T}}\right]$$

Thus $\pi_1 = \left(\dfrac{m\omega^2}{k}\right)$ $\qquad \pi_2 = \left(\dfrac{b\omega}{k}\right)$

6. (Continued)

e. Note that $\bar{\omega} = \pi_1^{\frac{1}{2}} = \frac{\omega m^{\frac{1}{2}}}{k^{\frac{1}{2}}}$ ← dimensionless frequency

f. i) But $\pi_2 = \frac{b\omega}{k}$ is not the same as $\bar{b} = \left(\frac{b^2}{4km}\right)^{\frac{1}{2}}$

ii) However, we can take

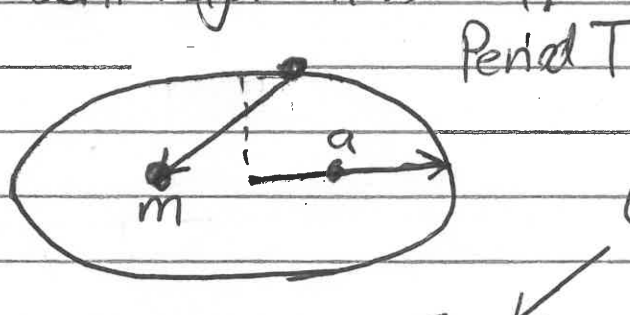
$$\pi_2 \pi_1^{-\frac{1}{2}} = \left(\frac{b\omega}{k}\right) \frac{k^{\frac{1}{2}}}{\omega m^{\frac{1}{2}}} = \left(\frac{b^2}{km}\right)^{\frac{1}{2}} = 2\bar{b}$$

g. Thus, we have recovered the fundamental dependencies

$$\boxed{\begin{aligned} \bar{\omega} &= \pi_1^{\frac{1}{2}} \\ \bar{b} &= \frac{1}{2}\pi_2 \pi_1^{-\frac{1}{2}} \end{aligned}}$$

B. Kepler's Third Law:

1. What is the relationship between the period T of an object orbiting elliptically about a mass m with a semi major axis is a?



universal constant of gravitation

2. Physical Quantities:

parameters	G	$\left[\frac{L^3}{MT^2}\right]$
	m	$[M]$
	a	$[L]$
dynamical quantity	T	$[T]$

I. B. (Continued)

3. $n=4$ physical quantities
   $k=3$ (M, L, T) fundamental units

   $m = n - k = 1$ dimensionless quantity

4. Create dimensionless quantity

$$\pi_1 = \frac{GmT^2}{a^3} \quad \left[\frac{\cancel{L^3}}{\cancel{M}\cancel{T^2}} \cancel{M}\cancel{T^2} \frac{1}{\cancel{L^3}}\right]$$

5. Thus $\phi(\pi_1) = 0 \Rightarrow \pi_1 - K = 0 \Rightarrow \pi_1 = K$

$$\Rightarrow \frac{GmT^2}{a^3} = K \Rightarrow T^2 = K\frac{a^3}{Gm}$$

$$\Rightarrow \boxed{T = \sqrt{K}\frac{a^{\frac{3}{2}}}{G^{\frac{1}{2}}m^{\frac{1}{2}}}} \qquad T \propto \frac{a^{\frac{3}{2}}}{m^{\frac{1}{2}}}$$

6. Analytical solution yields $K = 4\pi^2$, or

$$\boxed{T^2 = \frac{4\pi^2 a^3}{Gm}}$$

C. Pressure Drop along a Long, Cylindrical Pipe

1. What is the relation for the pressure gradient along a long, cylindrical pipe?

$\frac{dp}{dx}$

a. Experiments have shown the pressure gradient $\frac{dp}{dx}$ is constant along the pipe (not a function of position x along pipe).

2. Physical Quantities

pipe diameter	D^*	$[L]$
dynamic viscosity	μ	$[\frac{M}{LT}]$
mass density	ρ^*	$[M/L^3]$
flow velocity	U^*	$[L/T]$
pressure gradient	$\frac{dp}{dx}$	$[\frac{M}{L^2T^2}]$

a. Note: Dynamical vs. kinematic viscosity $\qquad \nu = \frac{\mu}{\rho}$

μ $[\frac{M}{LT}]$ $\qquad$ ν $[\frac{L^2}{T}]$

3. $n=5$ physical quantities
$k=3$ (M, L, T) fundamental units

$m = n-k = 2$ dimensionless variables

4. Create dimensionless quantities: For $k=3$, choose 3 quantities (*) the span the units (M, L, T). Make the other 2 dimensionless

II (4, Continued)

a. $\Pi_1 = \frac{dp}{dx}\frac{D}{\rho U^2}$ $\left[\frac{\cancel{M}}{\cancel{L^2}\cancel{T^2}}\frac{\cancel{L^3}}{\cancel{M}}\frac{\cancel{T^2}}{\cancel{L^2}}\cancel{L}\right]$

b. $\Pi_2 = \frac{\mu}{\rho U D}$ $\left[\frac{\cancel{M}}{\cancel{L}\cancel{T}}\frac{\cancel{L^3}}{\cancel{M}}\frac{\cancel{T}}{\cancel{L}}\frac{1}{\cancel{L}}\right]$

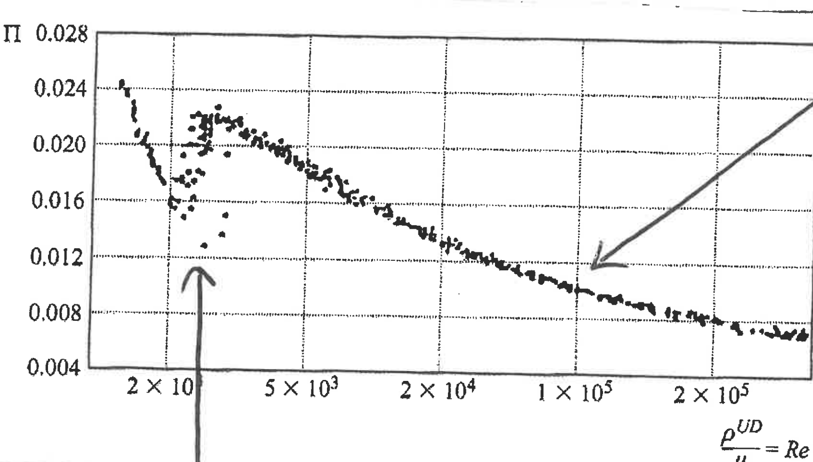
5. Thus, for two parameters, we obtain

$$\Pi_1 = f(\Pi_2) \Rightarrow \boxed{\frac{dp}{dx}\frac{D}{\rho U^2} = f\left(\frac{\mu}{\rho U D}\right)}$$

6. When experimental measurements are plotted as

$$\Pi_1 = \frac{dp}{dx}\frac{D}{\rho U^2} \quad \text{vs.} \quad Re = \Pi_2^{-1} = \frac{\rho U D}{\mu},$$

we obtain



all data falls on a single curve

laminar

turbulent

transition