Lecture #3 Perpendicular Waves and Physical Investigation of Wave Properties

I. Perpendicular Propagating Waves ($\underset{\sim}{k} \perp \underset{\sim}{B}$) (Continued)

A. 1. We saw last lecture there are two solutions.

a. $n^2 = P \Rightarrow$ Ordinary Mode (Modified Light Wave) (O)

b. $n^2 = \frac{RL}{S} \Rightarrow$ We'll see this is the Extraordinary Mode (X)

B. Extraordinary Mode: ($n^2 = \frac{RL}{S}$)

1. NOTE: $S - n^2 = S - \frac{RL}{S} = S - \frac{(S+D)(S-D)}{S} = \frac{1}{S}\left[S^2 - (S^2 - D^2)\right] = \frac{D^2}{S}$

a. This uses $R = S + D$ & $L = S - D$

b. Thus, we get

$$\begin{pmatrix} S & -iD & 0 \\ iD & \frac{D^2}{S} & 0 \\ 0 & 0 & P - \frac{RL}{S} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

2. Solving for Eigenfunction: a. Take $E_x = E_0$ & $E_z = 0$.

b. $S E_0 - iD E_y = 0 \Rightarrow E_y = \frac{-iS}{D} E_0$

c. So, the eigenfunction is $\underset{\sim}{E}_1 = \left(E_0, \frac{-iS}{D} E_0, 0\right)$

3. Both Longitudinal and Transverse Components:

a. NOTE: $\underset{\sim}{k} \times \underset{\sim}{E}_1 = 0$ implies longitudinal fluctuation

$\underset{\sim}{k} \cdot \underset{\sim}{E}_1 = 0$ implies transverse fluctuation.

b. For perpendicular propagation let's take $\underset{\sim}{k} = k \hat{\underset{\sim}{x}}$

c. Thus $\underset{\sim}{k} \times \underset{\sim}{E}_1 = k\hat{\underset{\sim}{x}} \times \left(E_0 \hat{\underset{\sim}{x}} - \frac{iS}{D} E_0 \hat{\underset{\sim}{y}}\right) = -\frac{iS}{D} k E_0 \hat{\underset{\sim}{z}} \neq 0$

$\underset{\sim}{k} \cdot \underset{\sim}{E}_1 = k\hat{\underset{\sim}{x}} \cdot \left(E_0 \hat{\underset{\sim}{x}} - \frac{iS}{D} E_0 \hat{\underset{\sim}{y}}\right) = k E_0 \neq 0$

d. This mode is more complicated with both longitudinal and transverse components.

I. B. (Continued)

4. Solution for Frequency: $n^2 = \frac{RL}{S}$

a. After some algebra, this can be written

$$\frac{k^2c^2}{\omega^2} = \frac{(\omega^2-\omega_R^2)(\omega^2-\omega_L^2)}{(\omega^2-\omega_{UH}^2)(\omega^2-\omega_{LH}^2)}$$

where, as before, $\omega_L = \frac{-|\omega_{ce}|}{2} + \frac{1}{2}\sqrt{\omega_{ce}^2 + 4(\omega_{pe}^2 + |\omega_{ce}|\omega_{ci})}$

$$\omega_R = \frac{|\omega_{ce}|}{2} + \frac{1}{2}\sqrt{\omega_{ce}^2 + 4(\omega_{pe}^2 + |\omega_{ce}|\omega_{ci})}$$

(Giving $\omega_R = \omega_L + |\omega_{ce}|$. NOTE ω_R, ω_L are solutions of $\omega^2 \mp |\omega_{ce}|\omega - \omega_{pe}^2 - |\omega_{ce}|\omega_{ci} = 0$)

b. We have defined the new frequencies:

NOTE: $\omega_{UH} > \omega_{LH}$ and $\omega_{UH} > \omega_L$

DEF: Upper hybrid frequency: $\omega_{UH}^2 = \omega_{pe}^2 + \omega_{ce}^2$

DEF: Lower hybrid frequency: $\omega_{LH}^2 = \omega_{ci}|\omega_{ce}| \frac{\omega_{pi}^2}{\omega_{pi}^2 + \omega_{ci}|\omega_{ce}|}$

where, for $\omega_{pi}^2 \gg \omega_{ci}|\omega_{ce}|$, $\omega_{LH}^2 \approx \omega_{ci}|\omega_{ce}|$

c. NOTE: In these definitions, we have made use of $\omega_{pe}^2 > \omega_{ce}^2 \gg \omega_{ci}^2$ to simplify.

5. Cutoffs & Resonances of Extraordinary mode $\quad \omega_R > \omega_{UH} > \omega_L > \omega_{LH}$

a. $n^2 > 0$ when $\omega > \omega_R$, $\omega_L < \omega < \omega_{UH}$, $\omega < \omega_{LH}$

$n^2 < 0$ (evanescent) when $\omega_{UH} < \omega < \omega_R$, $\omega_{LH} < \omega < \omega_L$

b. Resonances at: ($n^2 \to \infty$)

$\omega = \omega_{UH}$ Upper hybrid resonance

$\omega = \omega_{LH}$ Lower hybrid resonance

c. Cutoffs at ($n^2 \to 0$)

$\omega = \omega_R$

$\omega = \omega_L$

Lecture #3 (Continued)

I. (Continued)

C. Summary of Perpendicular Modes

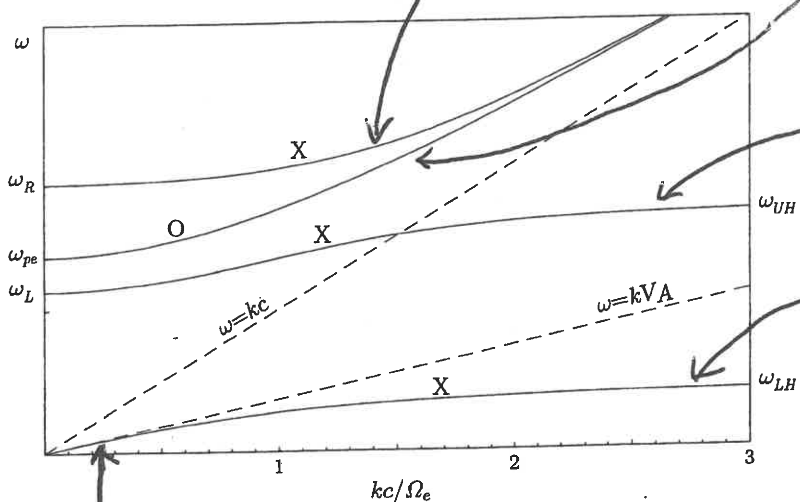
1. $\omega(k)$ for modes with perpendicular $\underline{k} = k\hat{x}$

Extraordinary Mode

Ordinary Mode

Upper hybrid Wave

Lower hybrid Wave



Magnetoacoustic (Fast) Wave

II. Summary of Parallel and Perpendicular Waves in a Cold, Magnetized Plasma

A. Parallel

1. **$P=0$:** $\boxed{\text{Plasma Oscillation}}$, longitudinal, motion $\parallel$ to $\mathbf{B}$

2. **$n^2=R$:** RH Circularly Polarized Transverse Wave
   $n^2=L$: LH Circularly Polarized Transverse Wave

Limits:
- a. $\boxed{\text{Alfvén Waves}}$ (L, R) $\omega \ll \omega_{ci} \ll |\omega_{ce}| < \omega_{pe}$
- b. $\boxed{\text{Ion Cyclotron Waves}}$ (L) $\omega \simeq \omega_{ci}$
- c. $\boxed{\text{Whistler Waves}}$ (R) $\omega_{ci} \ll \omega \ll |\omega_{ce}|$
- d. $\boxed{\text{Electron Cyclotron Waves}}$ (R) $\omega \simeq |\omega_{ce}|$
- e. $\boxed{\text{LH \& RH Circularly Polarized Modified Light Waves}}$ (L, R) $\omega^2 \simeq k^2c^2 + \omega_{pe}^2$

B. Perpendicular

1. **$n^2=P$:** $\boxed{\text{Ordinary Mode}}$, Modified Light Wave $\omega^2 = c^2k^2 + \omega_{pe}^2$

2. **$n^2=\frac{RL}{S}$:** $\boxed{\text{Extraordinary Mode}}$, longitudinal & transverse

Limits:
- a. $\boxed{\text{Magnetoacoustic (Fast) Wave}}$ $\omega \ll \omega_{LH}$
- b. $\boxed{\text{Lower Hybrid Waves}}$ $\omega \simeq \omega_{LH}$
- c. $\boxed{\text{Upper Hybrid Waves}}$ $\omega \simeq \omega_{UH}$
- d. $\boxed{\text{Extraordinary Mode Modified Light Wave}}$ $\omega^2 \simeq k^2c^2 + \omega_R^2$

Lecture #3 (Continued)

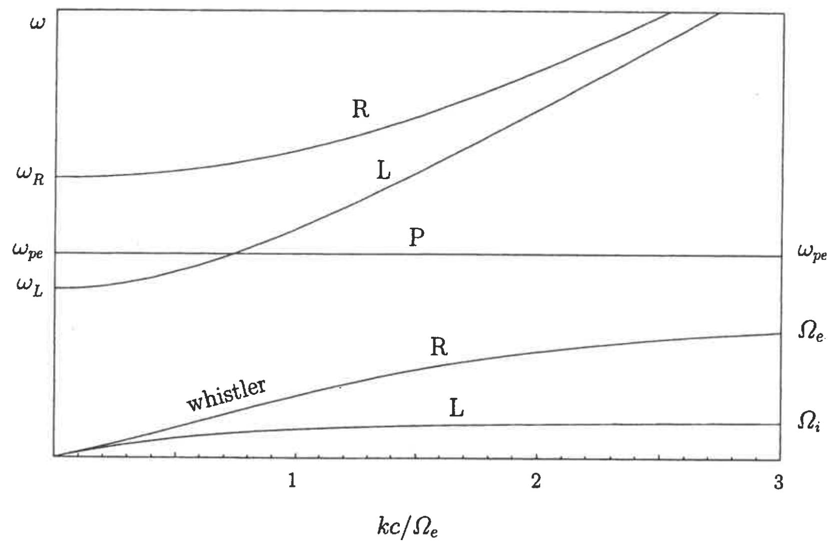
Cold Magnetized Plasma Waves for

III. Oblique Wavevectors: $0° < \theta < 90°$

Ref: Hazeltine & Waelbroeck, The Framework of Plasma Physics, 1998.

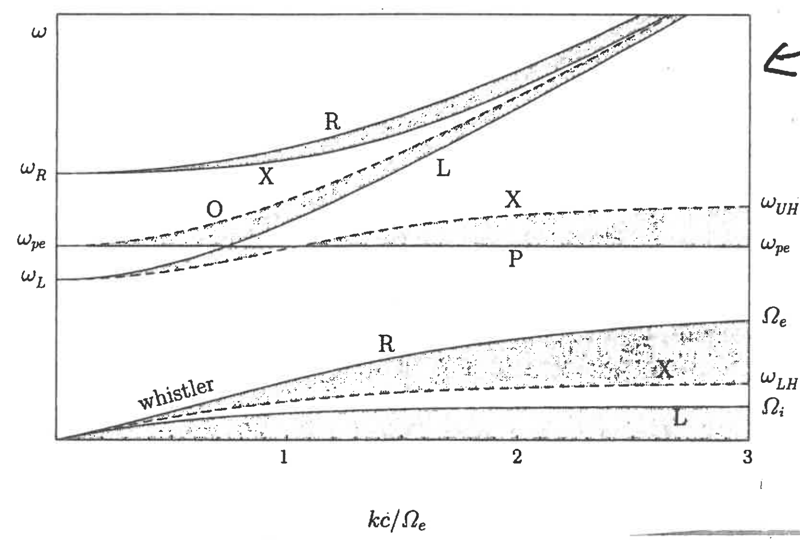
Parallel

$\Theta = 0°$



Oblique

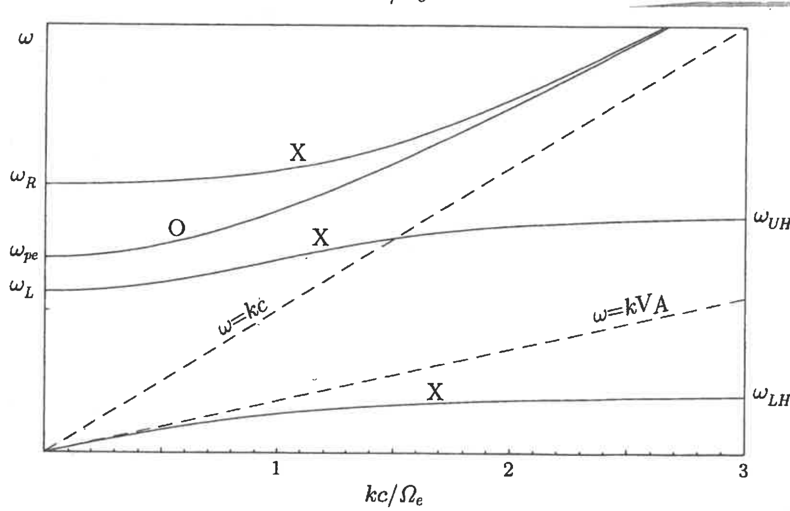
$0° < \Theta < 90°$



~~Shows~~ connection between parallel and perpendicular wave modes.

Perpendicular

$\Theta = 90°$



IV. The Lower Hybrid Wave

A. Limits of the Lower Hybrid Wave ($\underset{\sim}{k} \perp \underset{\sim}{B}_0$)

1. Take $\underset{\sim}{k} = k\hat{x}$ $\quad \underset{\sim}{B}_0 = B_0 \hat{\underset{\sim}{z}}$ (we can also write $\hat{b} = \frac{\underset{\sim}{B}}{B} = \hat{\underset{\sim}{z}}$
2. This Extraordinary Mode has $\underset{\sim}{E}_1 = \left(E_0, -\frac{iS}{D}E_0, 0\right)$
3. $\omega_{LH}^2 = \omega_{ci}|\omega_{ce}|$ (assuming $\omega_{pi}^2 \gg \omega_{ci}|\omega_{ce}|$)
4. We'll investigate the wave behavior in the limit $k \to \infty$.
5. For simplicity, assume a proton & electron plasma: $q_e = -q_i$, $\frac{m_i}{m_e} = 1836$

B. Ion Current:

1. Eq. of Motion (Momentum Eq.): $\underset{\sim}{u}_i = \underbrace{\frac{q_i}{-i\omega m_i}\underset{\sim}{E}_1}_{\text{Electric field term}} + \underbrace{\frac{q_i B_0}{-i\omega m_i}\underset{\sim}{u}_i \times \hat{\underset{\sim}{b}}}_{\text{Lorentz Force Term}}$

2. $\underset{\textcircled{1}}{\underset{\sim}{u}_i} = \underset{\textcircled{2}}{i\frac{q_i}{\omega m_i}\underset{\sim}{E}_1} + \underset{\textcircled{3}}{i\frac{\omega_{ci}}{\omega}\underset{\sim}{u}_i \times \hat{\underset{\sim}{b}}}$

3a. Let's compare the magnitudes of $\frac{\textcircled{3}}{\textcircled{1}}$ for $\omega = \omega_{LH}$

b. NOTE: For Lower Hybrid $\omega^2 = \omega_{ci}|\omega_{ce}| = \frac{q_i B_0}{m_i}\frac{|q_e|B_0}{m_e} = \frac{q_i^2 B_0^2}{m_i^2}\frac{m_i}{m_e} = \omega_{ci}^2 \frac{m_i}{m_e}$

Thus, $\frac{\omega_{ci}}{\omega} = \sqrt{\frac{m_e}{m_i}} \ll 1$

c. Thus $\mathcal{O}\left(\frac{\textcircled{3}}{\textcircled{1}}\right) = \frac{\frac{\omega_{ci}}{\omega}\cancel{u_i}}{\cancel{u_i}} \sim \frac{\omega_{ci}}{\omega} \ll 1$

$\Rightarrow$ Lorentz Force term ③ is negligible: **Ions are unmagnetized**

4. Ion current: $\underset{\sim}{j}_i = n_i q_i \underset{\sim}{u}_i = \frac{i\epsilon_0}{\omega}\frac{n_i q_i^2}{\epsilon_0 m_i}\underset{\sim}{E}_1 = \boxed{i\epsilon_0\frac{\omega_{pi}^2}{\omega}\underset{\sim}{E}_1 = \underset{\sim}{j}_i}$

C. Electron Current:

1. Eq. of Motion: $\underset{\sim}{u}_e = \frac{q_e}{-i\omega m_e}\underset{\sim}{E}_1 + \frac{q_e B_0}{-i\omega m_e}\underset{\sim}{u}_e \times \hat{\underset{\sim}{b}}$

2. $\underset{\textcircled{1}}{\underset{\sim}{u}_e} = \underset{\textcircled{2}}{i\frac{q_e}{\omega m_e}\underset{\sim}{E}_1} + \underset{\textcircled{3}}{i\frac{\omega_{ce}}{\omega}\underset{\sim}{u}_e \times \hat{\underset{\sim}{b}}}$

3. NOTE: $\omega^2 = \cancel{\omega_{ci}}|\omega_{ce}|^2 = \frac{q_e^2 B_0^2}{m_e^2}\frac{m_e}{m_i} \Rightarrow \frac{\omega_{ce}}{\omega} = \sqrt{\frac{m_i}{m_e}} \gg 1.$

4. Lorentz Force Dominates: **Electrons are magnetized!**

$\Rightarrow$ Larmor motion is lowest order motion

IV C. (Continued)

5. We can use our knowledge of single particle motion ⇒ drifts.

$$\underset{\sim}{u}_e = \underbrace{\frac{i q_e}{\omega m_e} E_z \hat{\underset{\sim}{b}}}_{\substack{\text{acceleration}\\ \text{along } \underset{\sim}{B} \text{ field}}} + \underbrace{\frac{\underset{\sim}{E}_1 \times \hat{\underset{\sim}{b}}}{B_0}}_{\substack{\underset{\sim}{E}\times\underset{\sim}{B} \text{ drift}\\ \text{(Lect \#3 PHYS:4731)}}} - \underbrace{\frac{i\omega}{\omega_{ce} B_0} \underset{\sim}{E}_1}_{\substack{\text{Polarization Drift}\\ \text{(Lect \#9 PHYS:4731)}}}$$

$E_z = 0$ for Lower Hybrid

6. Electron current: $\underset{\sim}{j}_e = n_e q_e \underset{\sim}{u}_e = \epsilon_0 \left(\frac{n_e q_e^2}{\epsilon_0 m_e}\right)\left(\frac{m_e}{q_e B_0}\right) \underset{\sim}{E}_1 \times \hat{\underset{\sim}{b}} - \frac{i\epsilon_0 \omega}{\omega_{ce}}\left(\frac{n_e q_e^2}{\epsilon_0 m_e}\right)\left(\frac{m_e}{q_e B_0}\right)\underset{\sim}{E}_1$

$$\boxed{\underset{\sim}{j}_e = \epsilon_0 \frac{\omega_{pe}^2}{\omega_{ce}} \underset{\sim}{E}_1 \times \underset{\sim}{b} - i\epsilon_0 \frac{\omega\,\omega_{pe}^2}{\omega_{ce}^2} \underset{\sim}{E}_1}$$

D. Limiting Behavior of Lower Hybrid as $k \to \infty$

1. Faraday's Law: $\underset{\sim}{k} \times \underset{\sim}{E}_1 = \omega \underset{\sim}{B}_1$ (A)

2. Ampere/Maxwell Law: $\underset{\sim}{k} \times \underset{\sim}{B}_1 = -i\mu_0 \underset{\sim}{j} - \frac{\omega}{c^2} \underset{\sim}{E}_1$

$$\Rightarrow c^2 \underset{\sim}{k} \times \underset{\sim}{B}_1 = \underbrace{\frac{\omega_{pi}^2}{\omega} \underset{\sim}{E}_1}_{\text{ion current}} - \underbrace{i \frac{\omega_{pe}^2}{\omega_{ce}} \underset{\sim}{E}_1 \times \hat{\underset{\sim}{b}}}_{\substack{\underset{\sim}{E}\times\underset{\sim}{B} \text{ electron}\\ \text{current}}} - \underbrace{\frac{\omega\,\omega_{pe}^2}{\omega_{ce}^2} \underset{\sim}{E}_1}_{\substack{\text{electron polarization}\\ \text{current}}} - \underbrace{\omega \underset{\sim}{E}_1}_{\substack{\text{displacement}\\ \text{current}}} \quad \text{(B)}$$

3. Lowest order Solution for $k \to \infty$

a. From (A) $\quad \underset{\sim}{k} \times \underset{\sim}{E}_{1(0)} = 0 \quad \Rightarrow \quad \underset{\sim}{E}_{1(0)} = E_0 \hat{\underset{\sim}{k}}$ $\boxed{\text{Electrostatic}}$

NOTE: This can also be seen from eigenfunction: $\underset{\sim}{E}_1 = (E_0, \frac{-iS}{D} E_0, 0)$

As $k \to \infty$, $S \to 1$ $D \to \infty$, so $\frac{E_y}{E_x} \to 0$.

b. From (B) $\quad \underset{\sim}{k} \times \underset{\sim}{B}_{1(0)} = 0 \quad \Rightarrow \underset{\sim}{B}_{1(0)} = B_1 \hat{\underset{\sim}{k}}$. But $\underset{\sim}{k} \cdot \underset{\sim}{B}_1 = 0$, so $B_1 = 0$! No magnetic response!

4. Next order: Using (B) with $\underset{\sim}{E}_{1(0)} = E_0 \hat{\underset{\sim}{k}}$, we get

$$c^2 \underset{\sim}{k} \times \underset{\sim}{B}_{1(1)} = \frac{\omega_{pi}^2}{\omega} E_0 \hat{\underset{\sim}{k}} - i \frac{\omega_{pe}^2}{\omega_{ce}} E_0 \hat{\underset{\sim}{k}} \times \hat{\underset{\sim}{b}} - \frac{\omega\,\omega_{pe}^2}{\omega_{ce}^2} E_0 \hat{\underset{\sim}{k}} - \omega E_0 \hat{\underset{\sim}{k}}$$

IV. D. (Continued)

5. Take $\hat{k}\cdot$ (B) to eliminate $B_{1(1)}$:

a. $0 = \frac{\omega_{pi}^2}{\omega}E_0 - \frac{\omega\,\omega_{pe}^2}{\omega_{ce}^2}E_0 - \omega E_0$

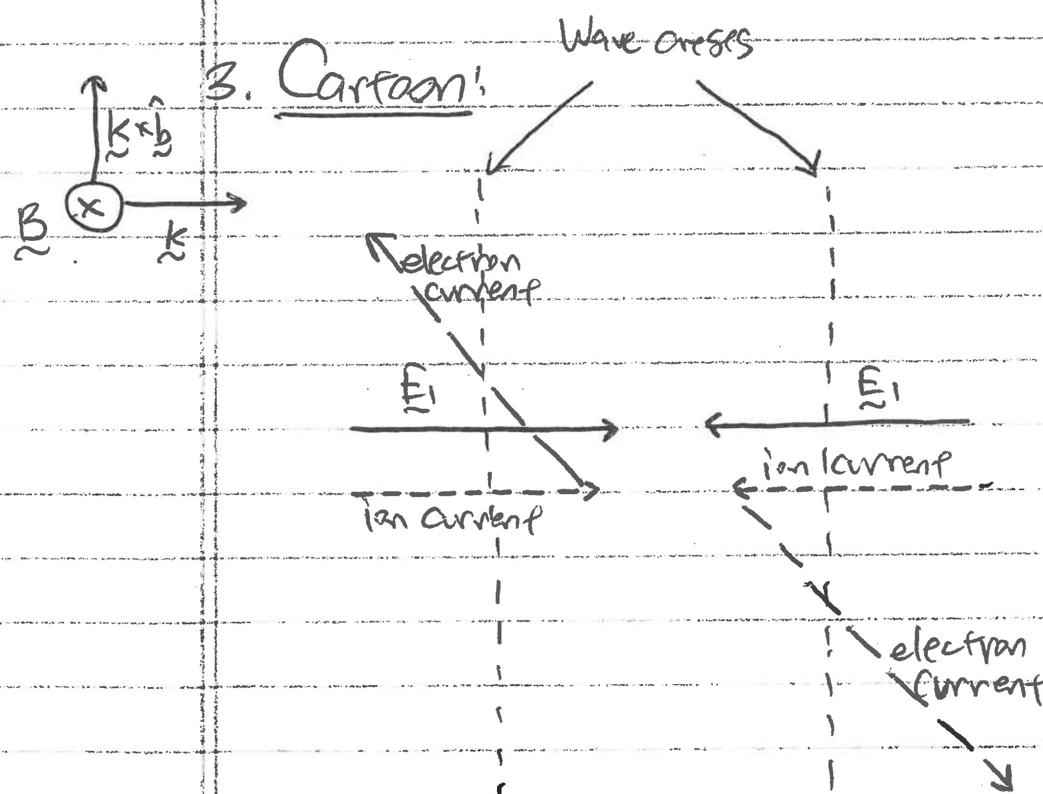
b. This can be solved to yield the Lower Hybrid Frequency

$$\boxed{\omega^2 = \omega_{ci}|\omega_{ce}|\left(\frac{\omega_{pi}^2}{\omega_{pi}^2+\omega_{ci}|\omega_{ce}|}\right)}$$

E. Physics of the Lower Hybrid Wave

1. The current along $\hat{\underset{\sim}{k}}$ ($\perp \underset{\sim}{B}_0$) sums to zero
   a. This is from i. ion current (unmagnetized)
      ii. electron polarization current
      iii. displacement current
2. $\underset{\sim}{E}\times\underset{\sim}{B}$ electron current suppresses (small) B_1 fluctuations.

3. Cartoon:



No charge build up:
$\nabla\cdot\underset{\sim}{j} = 0$
$\underset{\sim}{k}\cdot\underset{\sim}{j} = 0$ currents in $\hat{\underset{\sim}{k}}$ direction cancel.