8. Ordinary Mode light wave

① Limits of the waves ($\underset{\sim}{k} \perp \underset{\sim}{B_0}$)

1. Takes $\underset{\sim}{k} = k\hat{\underset{\sim}{x}}$ $\quad \underset{\sim}{B_0} = B_0 \hat{\underset{\sim}{z}}$ $\quad \left(\hat{\underset{\sim}{b}} = \frac{\underset{\sim}{B_0}}{B}\right)$

2. This Ordinary mode has $\underset{\sim}{E_1} = (0, 0, E_0)$

3. $$\omega^2 = k^2c^2 + \omega_p^2$$

4. We will investigate the wave behavior in the limit $k \to \infty$

5. For simplicity, assume a proton & electron plasma

$$q_e = -q_i \ , \quad \frac{m_i}{m_e} = 1836$$

B. Ion current:

1. Equation of Motion (momentum Eq.)

$$\underset{\sim}{U_i} = \underbrace{\frac{q_i}{-i\omega m_i} \underset{\sim}{E}}_{\substack{\text{Electric} \\ \text{field} \\ \text{term}}} + \underbrace{\frac{q_i B_0}{-i\omega m_i} \underset{\sim}{U_i} \times \hat{\underset{\sim}{b}}}_{\text{Lorentz force term.}}$$

2. $$\underset{①}{\underset{\sim}{U_i}} = \underset{②}{i \frac{q_i}{\omega m_i} \underset{\sim}{E_1}} + \underset{③}{i \frac{\omega_{ci}}{\omega} \underset{\sim}{U_i} \times \hat{\underset{\sim}{b}}} \qquad \left(\because \omega_{ci} = \frac{q_i B_0}{m_i}\right)$$

3. a. Let's compare the magnitude of $\frac{③}{①}$

b. NOTE: For ordinary lower mode: $\omega^2 = k^2c^2 + \omega_p^2$

$$\omega_p^2 = \omega_{pi}^2 + \omega_{pe}^2 \simeq \omega_{pe}^2$$

$$\Rightarrow \omega^2 = k^2c^2 + \omega_{pe}^2$$

given $\omega > \omega_{pe} \Rightarrow \omega \simeq kc$

thus, $\frac{\omega_{ci}}{\omega} \ll 1 \quad (\because k \to \infty)$

→ magnitudes

c. thus, $\Theta\left(\frac{③}{①}\right) = \frac{\frac{\omega_{ci}}{\omega}\cancel{v_i}}{\cancel{v_i}} = \frac{\omega_{ci}}{\omega} \ll 1$

→ Lorentz force term ③ is negligible $\boxed{\text{Ions are Unmagnetized}}$

4. Ion Current $J_i = n_i q_i \underset{\sim}{v_i}$

$$= n_i q_i \left(\frac{i q_i E_1}{\omega m_i}\right) \times \frac{\epsilon_0}{\epsilon_0}$$

$$\boxed{J_i = i\epsilon_0 \frac{\omega_{p_i}^2}{\omega} \underset{\sim}{E_1}}$$

$\left(\because \omega_{p_i}^2 = \frac{n_i q_i^2}{\epsilon_0 m_i}\right)$

C. Electron current:

1. Eq. of Motion: $\underset{\sim}{v_e} = -\frac{q_e}{-i\omega m_e}\underset{\sim}{E_1} + \frac{q_e B_0}{-i\omega m_e}\underset{\sim}{v_e} \times \underset{\sim}{\hat{b}}$

2. $\underset{\sim}{v_e} = \frac{i q_e}{\omega m_e}\underset{\sim}{E_1} + i\frac{\omega_{ce}}{\omega}\underset{\sim}{v_e} \times \underset{\sim}{\hat{b}}$ $\left(\omega_{c_e} = \frac{q_e B_0}{m_e}\right)$

① ② ③

3. NOTE: $\omega^2 = k^2c^2 + \omega_p^2$ $(\omega_p^2 \simeq \omega_{p_e}^2)$

given $\omega > \omega_{pe} \Rightarrow \omega \simeq kc$

thus, $\frac{\omega_{ce}}{\omega} \ll 1$ $(\because k \to \infty)$

thus magnitude $\Theta\left(\frac{③}{①}\right) = \frac{\frac{\omega_{ce}}{\omega}\cancel{v_e}}{\cancel{v_e}} = \frac{\omega_{ce}}{\omega} \ll 1$

→ Lorentz force term ③ is negligible

$\boxed{\text{electrons are Unmagnetized}}$

4. electron current $\underset{\sim}{J}_e = n_e q_e \underset{\sim}{v}_e$

$$= n_e q_e \left(\frac{i q_e}{\omega m_e}\right) \times \frac{\epsilon_0}{\epsilon_0}$$

$$\boxed{\underset{\sim}{J}_e = i \epsilon \frac{\omega_{pe}^2}{\omega} \underset{\sim}{E}_1} \qquad \left(\because \omega_{pe}^2 = \frac{n_e q_e^2}{\epsilon_0 m_e}\right)$$

D Limiting Behavior of ordinary mode light wave

1 Faraday's Law : $\underset{\sim}{k} \times \underset{\sim}{E}_1 = \omega \underset{\sim}{B}_0$ (A)

2 Ampere/Maxwell Law : $\underset{\sim}{k} \times \underset{\sim}{B}_0 = -i \mu_0 J - \frac{\omega}{c^2} \underset{\sim}{E}_1$

$$\Rightarrow c^2 \underset{\sim}{k} \times \underset{\sim}{B}_1 = \underbrace{\frac{\omega_{pi}^2}{\omega} \underset{\sim}{E}_1}_{\text{ion current}} + \underbrace{\frac{\omega_{pe}^2}{\omega} \underset{\sim}{E}_1}_{\text{electron current}} - \underbrace{\omega \underset{\sim}{E}_1}_{\text{displacement current}} \quad \text{(B)} \qquad \left(\mu_0 \epsilon_0 = \frac{1}{c^2}\right)$$

3 Lowest order solution for $k \to \sim$

a. From (A) $\underset{\sim}{k} \times \underset{\sim}{E}_1 \neq 0$ $\quad \underset{\sim}{E}_1 = E_0 \hat{z}$ and $\underset{\sim}{k} = k \hat{x}$

so $\underset{\sim}{B}_1 = \frac{1}{\omega} (\underset{\sim}{k} \times \underset{\sim}{E})$

L.H.S

from (B) $= \frac{1}{\omega} \underbrace{\underset{\sim}{k} \times (\underset{\sim}{k} \times \underset{\sim}{E}_1)}$

$$\underset{\sim}{k} \times (\underset{\sim}{k} \times \underset{\sim}{E}) = \underset{\sim}{k}(\underset{\sim}{k} \cdot \underset{\sim}{E}) - \underset{\sim}{E}(\underset{\sim}{k} \cdot \underset{\sim}{k})$$

$$= -k^2 \underset{\sim}{E}$$

NRL formula

(..b..)

$= b(\underset{\sim}{a} \cdot \underset{\sim}{c}) - (\underset{\sim}{a} \cdot \underset{\sim}{b})$

$$-\frac{c^2 k^2}{\omega} \underset{\sim}{E} = \frac{\omega_{pi}^2}{\omega} \underset{\sim}{E}_1 + \frac{\omega_{pe}^2}{\omega} - \omega \underset{\sim}{E}_1$$

4

$$\omega^2 = c^2k^2 + \omega_p^2 \qquad \left(\because \omega_p^2 = \omega_{p_i}^2 + \omega_{p_e}^2 \right)$$

$k \to \infty \qquad \omega \approx ck \qquad \text{and } \frac{\omega}{\omega_{ci}} \gg 1$

E. Physics of the ordinary mode light waves.

1. The current is along $\hat{z}$ ($\parallel \underset{\sim}{B_0}$), so mag. field has no effects on waves.

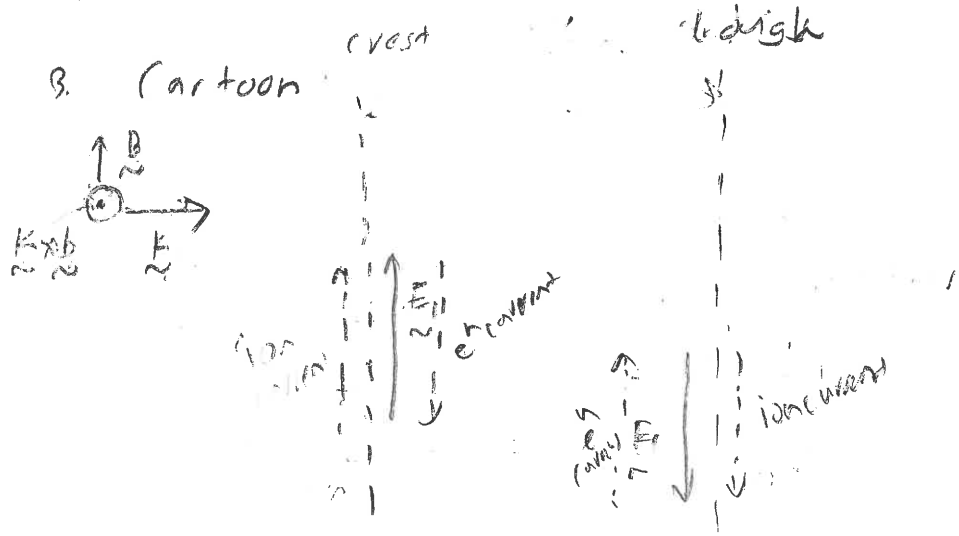
(a.) This form (i) ion current (unmagnetized)

(ii) electron current (unmagnetized)

(iii) displacement current

At high k, the wave behaves like a standard light wave in vacuum.

B. Cartoon



if $\frac{\partial E}{\partial t}$ is small.

Joshua Doucette

#1.) $\theta = 0$, $\vec{k} = k\hat{z}$, $\vec{B}_0 = B_0\hat{z}$

$$\vec{E}_1 = (E_0, iE_0, 0)$$

We want to find behavior in $k \to \infty$ limit

#2.)

$$\vec{u}_i = \frac{q_i \vec{E}_1}{-i\omega M_i} + \frac{q_i B_0}{-i\omega M_i}(\vec{u}_i \times \hat{b})$$

(1) (2) (3)

$$\frac{(3)}{(1)} \sim \frac{\omega_{ci}}{\omega_{ce}} \sim \frac{m_e}{M_i} \sim \frac{1}{1836}$$

$\Rightarrow$ ions are unmagnetized

$$\Rightarrow \vec{j}_i = i\epsilon_0\left(\frac{\omega_{pi}^2}{\omega}\right)\vec{E}_1$$

At $\omega \sim \omega_{ce}$: Drift limit is not valid

(4.4.4)

$$\begin{pmatrix} u_{ex} \\ u_{ey} \\ u_{ez} \end{pmatrix} = \frac{e}{m_e}\begin{pmatrix} \left(\frac{-i\omega}{\omega_{ce}^2-\omega^2}\right) & \left(\frac{\omega_{ce}}{\omega_{ce}^2-\omega^2}\right) & 0 \\ \left(\frac{-\omega_{ce}}{\omega_{ce}^2-\omega^2}\right) & \left(\frac{-i\omega}{\omega_{ce}^2-\omega^2}\right) & 0 \\ 0 & 0 & \frac{i}{\omega} \end{pmatrix}\begin{pmatrix} E_{1x} \\ E_{1y} \\ E_{1z} \end{pmatrix}$$

$$u_{ex} = \left(\frac{e}{m_e}\right)\left[\left(\frac{-i\omega}{\omega_{ce}^2-\omega^2}\right)E_{1x} + \left(\frac{\omega_{ce}}{\omega_{ce}^2-\omega^2}\right)E_{1y}\right]$$

$$u_{ey} = \left(\frac{e}{m_e}\right)\left[\left(\frac{-\omega_{ce}}{\omega_{ce}^2-\omega^2}\right)E_{1x} + \left(\frac{-i\omega}{\omega_{ce}^2-\omega^2}\right)E_{1y}\right]$$

$$u_{ez} = 0$$

$$\Rightarrow J_{ex} = \epsilon_0 \omega_{pe}^2\left[\left(\frac{-i\omega}{\omega_{ce}^2-\omega^2}\right)E_{1x} + \left(\frac{\omega_{ce}}{\omega_{ce}^2-\omega^2}\right)E_{1y}\right]$$

$$J_{ey} = \epsilon_0 \omega_{pe}^2\left[\left(\frac{-\omega_{ce}}{\omega_{ce}^2-\omega^2}\right)E_{1x} + \left(\frac{-i\omega}{\omega_{ce}^2-\omega^2}\right)E_{1y}\right]$$

$$J_{ez} = 0$$

#3)

Faraday's law: $\vec{\nabla}\times\vec{E} = \frac{-\partial\vec{B}}{\partial t} \Rightarrow \vec{k}\times\vec{E}_1 = \omega\vec{B}_1$

Ampere's law: $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \Rightarrow \vec{k}\times\vec{B}_1 = -i\mu_0\vec{j} - \frac{\omega}{c^2}\vec{E}_1$

$$\Rightarrow \vec{B}_1 = \frac{\vec{k}\times\vec{E}_1}{\omega}$$

Ampere's Law : (Let $\gamma^2 = \frac{\omega^2}{c^2} - k^2$)

$$\frac{\vec{k} \times (\vec{k} \times \vec{E}_1)}{\omega} = -i\mu_0 \vec{j} - \frac{\omega}{c^2} \vec{E}_1$$

$$\vec{k} \times (\vec{k} \times \vec{E}_1) = (\vec{k} \cdot \vec{E}_1)\vec{k} - (\vec{k} \cdot \vec{k})\vec{E}_1$$

But $\vec{k} \cdot \vec{E}_1 = 0$

$$\Rightarrow \frac{-k^2 \vec{E}_1}{\omega} = -i\mu_0 \vec{j} - \frac{\omega}{c^2} \vec{E}_1$$

$$\Rightarrow \left(\frac{\omega^2}{c^2} - k^2\right)\vec{E}_1 = -i\mu_0 \omega \vec{j} = -i\mu_0 \omega [\vec{j}_i + \vec{j}_e]$$

$$\Rightarrow \gamma^2 \vec{E}_1 = \mu_0 \epsilon_0 \omega_{pi}^2 \vec{E}_1 - i\mu_0 \omega \vec{j}_e$$

Dot Both sides w/ $\hat{x}$:

$$\gamma^2 E_{1x} = \left(\frac{\omega_{pi}^2}{c^2}\right) E_{1x} - i\mu_0 \omega j_{ex}$$

Dot Both sides w/ $\hat{y}$:

$$\gamma^2 E_{1y} = \left(\frac{\omega_{pi}^2}{c^2}\right) E_{1y} - i\mu_0 \omega j_{ey}$$

$$\gamma^2 E_{1x} = \left(\frac{\omega_{pi}^2}{c^2}\right) E_{1x} - i\mu_0 \omega j_{ex}$$

$$\gamma^2 E_{1y} = \left(\frac{\omega_{pi}^2}{c^2}\right) E_{1y} - i\mu_0 \omega j_{ey}$$

$$\left[\gamma^2 - \left(\frac{\omega_{pi}^2}{c^2}\right)\right] E_{1x} = -i\mu_0 \omega j_{ex}$$

$$\left[\gamma^2 - \left(\frac{\omega_{pi}^2}{c^2}\right)\right] E_{1y} = -i\mu_0 \omega j_{ey}$$

$$\Rightarrow \left[\gamma^2 - \left(\frac{\omega_{pi}^2}{c^2}\right)\right]^2 |\vec{E}|^2 = -\mu_0^2 \omega^2 \left(j_{ex}^2 + j_{ey}^2\right)$$

$$j_{ex}^2 = \epsilon_0^2 \omega_{pe}^4 \omega^2 \left[\frac{-\omega^2}{(\omega_{ce}^2-\omega^2)^2} E_{1x}^2 - \frac{2i\omega\omega_{ce}}{(\omega_{ce}^2-\omega^2)^2} E_{1x}E_{1y} + \frac{\omega_{ce}^2}{(\omega_{ce}^2-\omega^2)^2} E_{1y}^2\right]$$

$$j_{ey}^2 = \epsilon_0^2 \omega_{pe}^4 \omega^2 \left[\frac{\omega_{ce}^2}{(\omega_{ce}^2-\omega^2)} E_{1x}^2 + \frac{2i\omega\omega_{ce}}{(\omega_{ce}^2-\omega^2)^2} E_{1x}E_{1y} - \frac{\omega^2}{(\omega_{ce}^2-\omega^2)^2} E_{1y}^2\right]$$

$$\Rightarrow \left[\gamma^2 - \left(\frac{\omega_{pi}^2}{c^2}\right)\right]^2 |\vec{E}|^2 = \frac{\mu_0^2 \epsilon_0^2 \omega_{pe}^4 \left[\omega^2 - \omega_{ce}^2\right] \omega^2}{\left[\omega_{ce}^2 - \omega^2\right]^2} |\vec{E}|^2$$

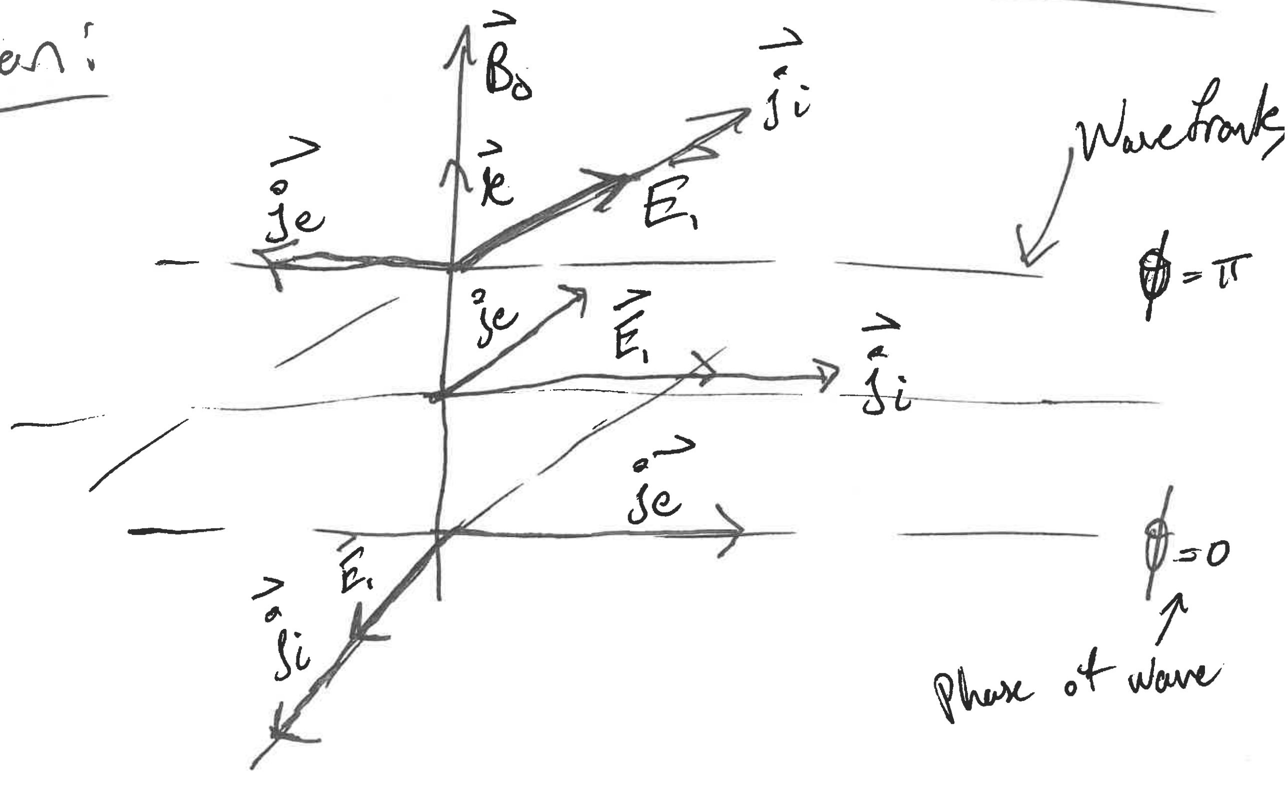
$$\left[\gamma^2 - \left(\frac{\omega_{pi}^2}{c^2}\right)\right]^2 = \frac{\omega_{pe}^4\,\omega^2}{c^4[\omega^2-\omega_{ce}^2]}$$

$$\left[c^2\gamma^2 - \omega_{pi}^2\right]^2 = \frac{\omega_{pe}^4\,\omega^2}{[\omega^2-\omega_{ce}^2]}$$

$$[\omega^2-\omega_{ce}^2] = \frac{\omega_{pe}^4\,\omega^2}{[\omega^2 - c^2k^2 - \omega_{pi}^2]^2} = \frac{\omega_{pe}^4}{\left[1-\frac{c^2k^2}{\omega^2}-\frac{\omega_{pi}^2}{\omega^2}\right.}$$

As $k \longrightarrow \infty$ $\boxed{\omega^2 \longrightarrow \omega_{ce}^2}$

Cartoon:



Brendan Powers Phys 7729 HW #1

Ion cyclotron wave solution to cold magnetized Plasma

First we consider the $n^2 = L$ root to the dispertion relation for $\vec{K} \parallel \vec{B}$ to see

$$\begin{bmatrix} D & -iD & 0 \\ iD & D & 0 \\ 0 & 0 & P \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \rightarrow \begin{matrix} DE_x - iDE_y = 0 \\ iDE_x + DE_y = 0 \\ PE_z = 0 \end{matrix}$$

if we choose $E_x = E_0$ $\boxed{\vec{E} = (E_0, -iE_0, 0)}$

Or with our definition of L we can immediately see

$$n^2 = L = 1 - \sum_s^{e,i} \frac{\omega_{ps}^2}{\omega(\omega - \omega_{cs})} \rightarrow \boxed{\frac{c^2k^2}{\omega^2} = \frac{(\omega - \omega_L)(\omega + \omega_R)}{(\omega - \omega_{ce})(\omega - \omega_{ci})}}$$

Since n must be real for the wave to exist we can show $0 < \omega < \omega_{ci}$ or $\omega > \omega_L$

We are only interested in this first solution

This tells us our frequency limits but nothing about the physics of the wave!!!

To understand the Physics, lets see how plasma responds to the wave Electric Field

remember $\vec{K} \parallel \vec{B}$ + $\vec{E}_1 = (E_0, -iE_0, 0)$

First, the Electron momentum

$$\underset{①}{\omega \vec{v}_e} = \underset{②}{i \frac{q_e}{m_e} \vec{E}_1} + \underset{③}{i\omega_{ce} \vec{v}_e \times \hat{z}}$$

see if one of the $\vec{v}_e$ terms in negligible in our limit $\omega \approx \omega_{ci}$

$$\frac{\omega_{ce}}{\omega_{ci}} = \frac{m_i}{m_e} \gg 1$$

Term ③ dominates! Electrons are dominated by the first order motion

Therefore

$$\boxed{J_e = n_e q_e \vec{v}_e = \varepsilon_0 \frac{\omega_{pe}^2}{\omega_{ce}} \vec{E}_1 \times \hat{b} - i\varepsilon_0 \frac{\omega\, \omega_{pe}^2}{\omega_{ce}^2}\vec{E}_1}$$

Ion Momentum

$$\underset{①}{\omega \vec{v}_i} = \underset{②}{i \frac{q_i}{m_i}\vec{E}_1} + \underset{③}{i\,\omega_{ci}\vec{v}_i\times\hat{b}}$$

Again compare terms ①+③ in our limit $\omega \to \omega_{ci}$

Solve the vector system of Eqs. $\frac{\omega_{ci}}{\omega} \to 1$ no terms are negligible

$$\boxed{J_i = i\varepsilon_0 \frac{\omega_{pi}}{\omega-\omega_{ci}} E_0\hat{x} + \varepsilon_0\frac{\omega_{pi}}{(\omega-\omega_{ci})}E_0\hat{y}}$$

If we plug in our derived ion & electron currents, Ampere's law becomes

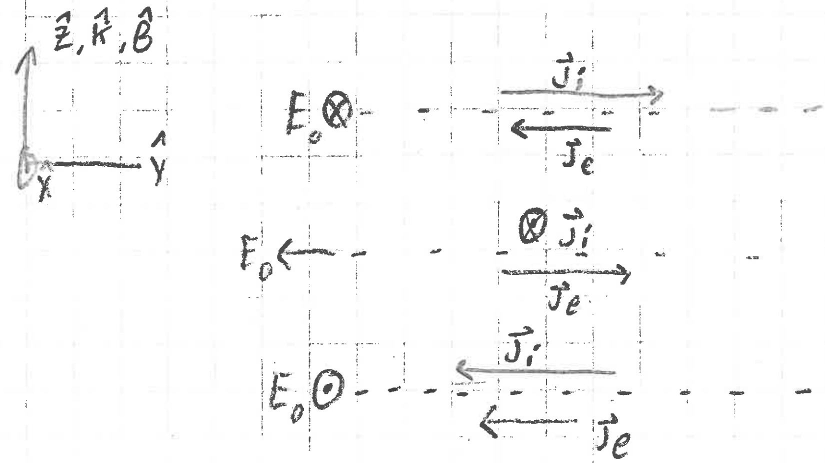
$$c^2\vec{k}\times\vec{B} = \frac{\omega_{pi}}{\omega-\omega_{ci}}\vec{E}_1 + i\overset{\approx 0}{\frac{\omega_{pe}^2}{\omega_{ce}}}\vec{E}_1\times\hat{b} + \overset{\approx 0}{\frac{\omega\,\omega_{pe}^2}{\omega_{ce}^2}}\vec{E}_1 - \omega\vec{E}_1$$

taking $\omega_{pe} > \omega_{ce}$

Then in the limit that $k \to \infty$ we can see that the denominator on the RHS goes to zero

$$\boxed{\omega \approx \omega_{ci}}$$

Visualizing the ion cyclotron wave



Yuanzheng Wen

A. Limits of the cold Plasma Alfvén wave $(\vec{k} // \vec{B}_0)$

1. Take $\vec{k} = k\hat{z}$, $\vec{B}_0 = B_0\hat{z}$

2. The Right & Left hand mode has $\vec{E}_1 = (E_0, \pm iE_0, 0)$

3. $\omega \ll \omega_{ci} \ll |\omega_{ce}| < \omega_{pe}$ 4. For simplicity, $q_e = -q_i$, $\frac{m_i}{m_e} = 1836$, $n_i = n_e$

B. Ion current

Eq of Motion: $\underset{①}{\vec{u}_i} = \underset{②}{\frac{iq_i}{\omega m_i}\vec{E}_1} + i\frac{q_i B_0}{\omega m_i}\vec{u}_i \times \hat{z} = \frac{iq_i}{\omega m_i}\vec{E}_1 + \underset{③}{i\frac{\omega_{ci}}{\omega}\vec{u}_i \times \hat{z}}$

compare the magnitude of ① and ③

$$\left.\begin{array}{l}\frac{③}{①} = \frac{\omega_{ci}}{\omega} \\ \text{since } \omega_{ci} \gg \omega\end{array}\right\} \Rightarrow \frac{\omega_{ci}}{\omega} \gg 1$$: Lorenze Force dominates, ions are magnitized

$$\vec{u}_i = \frac{iq_i}{\omega m_i}\vec{E}_z + \frac{\vec{E}_1 \times \hat{b}}{B_0} - \frac{i\omega}{\omega_{ci} B_0}\vec{E}_1$$

As Alfvén wave mode has no E_z, $E_z = 0$

$$\vec{j}_i = n_i q_i\left(\frac{\vec{E}_1 \times \hat{b}}{B_0} - \frac{i\omega}{\omega_{ci} B_0}\vec{E}_1\right) = \left(\frac{n_i q_i^2}{\epsilon_0 m_i}\right)\left(\frac{m_i}{q_i B_0}\right)\vec{E}_1 \times \hat{b} - i\omega\epsilon_0\left(\frac{n_i q_i^2}{\epsilon_0 m_i}\right)\left(\frac{m_i}{q_i B_0}\right)\vec{E}_1$$

$$= \frac{\omega_{pi}^2}{\omega_{ci}}\vec{E}_1 \times \hat{b} - i\epsilon_0\frac{\omega\,\omega_{pi}^2}{\omega_{ci}}\vec{E}_1$$

C. Electron current

$$\underset{①}{\vec{u}_e} = \underset{②}{\frac{iq_e}{\omega m_e}\vec{E}_1} + \underset{③}{i\frac{\omega_{ce}}{\omega}\vec{u}_e \times \hat{z}}$$

Following the same procedure for ion current

$\frac{③}{①} = \frac{\omega_{ce}}{\omega} \gg 1$, still the lorenze force dominates, Electrons are dominated

$$\vec{u}_e = \frac{iq_e}{\omega m_e}E_z\hat{b} + \frac{\vec{E}_1 \times \hat{b}}{B_0} - \frac{i\omega}{\omega_{ce} B_0}\vec{E}_1$$

since $\vec{E}_1$ doesn't have E_z component, thus $E_z = 0$

$$\vec{u}_e = \frac{\vec{E}_1 \times \hat{b}}{B_0} - \frac{i\omega}{\omega_{ce} B_0}\vec{E}_1 \qquad \omega_{pi}^2 = \frac{n_i q_i^2}{\varepsilon_0 m_i}, \quad \omega_{pe} = \frac{n_e q_e^2}{\varepsilon_0 m_e}$$

$$\vec{j}_e = n_e q_e \vec{u}_e = \epsilon_0 \left(\frac{n_e q_e^2}{\epsilon_0 m_e}\right)\left(\frac{m_e}{q_e B_0}\right)\vec{E}_1 \times \hat{b} - \frac{i\epsilon_0 \omega}{\omega_{ce}}\left(\frac{n_e q_e^2}{\epsilon_0 m_e}\right)\left(\frac{m_e}{q_e B_0}\right)\vec{E}_1$$

$$= \epsilon_0 \frac{\omega_{pe}^2}{\omega_{ce}} \vec{E}_1 \times \hat{b} - \frac{i\epsilon_0 \omega\, \omega_{pe}^2}{\omega_{ce}^2}\vec{E}_1$$

D. Limits Behavior of Alfven wave as $k \to 0$

1. Farady's Law: $\vec{k} \times \vec{E}_1 = \omega \vec{B}_1$
2. Ampere Law: $\vec{k} \times \vec{B}_1 = -i\mu_0 \vec{j} - \frac{\omega}{c^2}\vec{E}_1$

$$c^2 \vec{k} \times \vec{B} = -i \frac{1}{\cancel{\mu_0} \epsilon_0} \cancel{\mu_0} \left[\epsilon_0 \frac{\omega_{pe}^2}{\omega_{ce}} \vec{E}_1 \times \vec{b} - i\epsilon_0 \frac{\omega\, \omega_{pe}^2}{\omega_{ce}^2}\vec{E}_1 + \epsilon_0 \frac{\omega_{pi}^2}{\omega_{ci}} \vec{E}_1 \times \vec{b} - i\epsilon_0 \frac{\omega\, \omega_{pi}^2}{\omega_{ci}^2}\vec{E}_1\right] - \omega \vec{E}_1$$

$$= \underbrace{-i \frac{\omega_{pe}^2}{\omega_{ce}} \vec{E}_1 \times \vec{b}}_{\vec{E}\times\vec{B}\ e\ \text{current}} - \underbrace{\frac{\omega\, \omega_{pe}^2}{\omega_{ce}^2}\vec{E}_1}_{e\ \text{polarization}} \underbrace{- i\frac{\omega_{pi}^2}{\omega_{ci}} \vec{E}_1 \times \vec{b}}_{\vec{E}\times\vec{B}\ i\ \text{current}} - \underbrace{\frac{\omega\, \omega_{pi}^2}{\omega_{ci}^2}\vec{E}_1}_{i\ \text{polarization}} \underbrace{- \omega \vec{E}_1}_{\text{displacement current}}$$

$$\frac{\omega_{ps}^2}{\omega_{cs}} = \frac{n_s q_s^2}{\varepsilon_0 m_s}\cdot\frac{m_s}{q_s B_0} = \frac{n_s q_s}{\varepsilon_0 B_0}, \quad \frac{\omega_{ps}^2}{\omega_{cs}^2} = \frac{n_s q_s^2}{\varepsilon_0 m_s}\cdot\frac{m_s^2}{q_s^2 B_0^2} = \frac{n_s m_s}{\varepsilon_0 B_0^2}$$

$$= -i \frac{\cancel{n_i q_i}}{\varepsilon_0 B_0} \vec{E}_1 \times \vec{b} - \frac{i\, \cancel{n_e q_e}}{\varepsilon_0 B_0} \vec{E}_1 \times \vec{b} - \omega \frac{\omega_{pe}^2}{\omega_{ce}^2}\vec{E}_1 - \frac{\omega\, \omega_{pi}^2}{\omega_{ci}^2}\vec{E}_1 - \omega \vec{E}_1$$

$$\frac{c^2}{\omega}\vec{k} \times (\vec{k} \times \vec{E}_1) = \frac{c^2}{\omega}\left[k^2 E_0 (\hat{z} \times \hat{y}) \mp i k^2 E_0 (\hat{z} \times \hat{x})\right]$$

$$= \frac{c^2}{\omega}\left(-k^2 E_0 \hat{x} \mp i k^2 E_0 \hat{y}\right)$$

$$= \frac{-c^2}{\omega} k^2 \vec{E}_1$$

$$\frac{c^2}{\omega} k^2 \vec{E}_1 = \frac{\omega m_i n_i}{\varepsilon_0 B_0^2}\vec{E}_1 + \omega \frac{m_e n_e}{\varepsilon_0 B_0^2}\vec{E}_1 + \omega \vec{E}_1$$

$$\Rightarrow c^2k^2\vec{E}_1 = \omega^2\vec{E}_1\left(\frac{m_i n_i}{\varepsilon_0 B_0^2} + \frac{m_e n_e}{\varepsilon_0 B_0^2}\right)\vec{E}_1 + \omega^2\vec{E}_1$$

$$\Rightarrow c^2k^2\vec{E}_1 = \omega^2\vec{E}_1\left[\underset{(1)}{\frac{m_i n_i}{\varepsilon_0 B_0^2}} + \underset{(2)}{\frac{m_e n_e}{\varepsilon_0 B_0^2}} + \underset{(3)}{1}\right]$$

Since $n_e \ll n_i$, so the electron dipolarization current is negligible.

the ratio between displacement current and ion dipolarization current:

$$\frac{\omega\vec{E}_1}{\omega\frac{\omega_{pi}^2}{\omega_{ci}^2}\vec{E}_1} = \frac{\omega_{ci}^2}{\omega_{pi}^2} \ll 1$$

, thus displacement current is also negligible

$$c^2k^2\vec{E}_1 = \omega^2\vec{E}_1\cdot\frac{m_i n_i}{\varepsilon_0 B_0^2}$$

$$\Rightarrow \omega^2 = c^2k^2\cdot\frac{\varepsilon_0 B_0^2}{m_i n_i} = \frac{1}{\mu_0\varepsilon_0}k^2\frac{\varepsilon_0 B_0^2}{m_i n_i} = k^2\frac{B_0^2}{\mu_0(\underbrace{m_i n_i}_{\rho})} = k^2V_A^2$$

Physics of the cold Plasma Alfven wave

(1) The Sum of $\vec{E}\times\vec{B}$ drift Current (Ion & Electron) is Zer

(2) The displacement current and electron dipolarization current 0

(3) Ion dipolarization current

$$\vec{E}_1 = (E_0, \pm iE_0, 0)$$

Cartoon: For Right-hand $\quad \vec{E}(x,t) = E_0\hat{x}e^{i(kz-wt)} + iE_0\hat{y}e^{i(kz-wt)}$

Left-hand $\quad$ the real part at $z=0$ $\quad E = E_0\cos wt\,\hat{x} + E_0\sin(wt)\,\hat{y}$