Lecture #7 : Drift Waves

I. Drift Waves

A. General Comments

1. Our investigation of plasma waves thus far has focused on infinite, uniform plasmas with a straight magnetic field.
2. However, most plasmas about which we care are confined, and therefore have density gradients.
3. An important class of waves that exist only in plasmas with a density or temperature gradient are Drift Waves.

B. Drift Waves in a Plasma with a Density Gradient

1. Low Beta Plasma: $\frac{m_e}{m_i} \ll \beta_e \ll 1$ where $\beta_e \equiv \frac{2\mu_0 n_e T_e}{B_0^2}$

   a. Here magnetic pressure dominates over thermal pressure.

2. a. $\underset{\sim}{B}_0 = B_0 \hat{z}$ $\quad \underset{\sim}{E}_0 = 0$ Straight, Uniform $\underset{\sim}{B}_0$

   b. $n_{i0} = n_{e0} = n_0(x)$ Density Gradient

   c. $T_e = T_{e0} =$ constant Isothermal electrons.

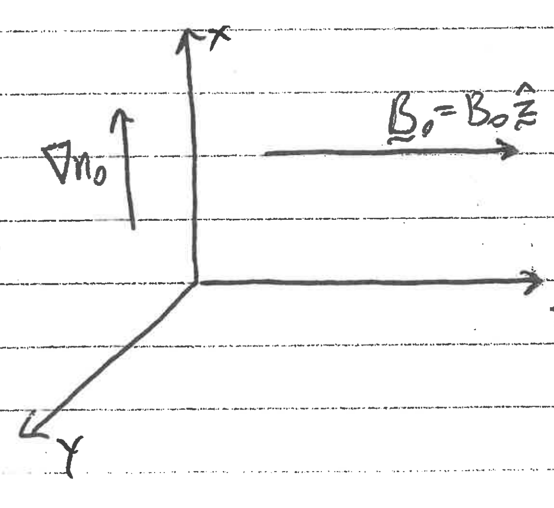
   Thus, the electron eq. of state is $\boxed{p_e = n_e T_e}$ $(\gamma_e = 1)$

   (NOTE: I have absorbed Boltzmann's constant k into T_e)

   d. $T_i = 0$ Cold Ions

In these limits, we will solve for Electron Drift Waves

3. Geometry:



I. B. (Continued)

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4. Two-Fluid Treatment: (See Lecture #14, III.)

a. In this limit, the two fluid system is

Continuity: $\frac{\partial n_i}{\partial t} + \nabla\cdot(n_i \underset{\sim}{U_i}) = 0$ $\qquad$ $\frac{\partial n_e}{\partial t} + \nabla\cdot(n_e \underset{\sim}{U_e}) = 0$

Momentum: $m_i n_i \left[\frac{\partial \underset{\sim}{U_i}}{\partial t} + \underset{\sim}{U_i}\cdot\nabla \underset{\sim}{U_i}\right] = q_i n_i (\underset{\sim}{E} + \underset{\sim}{U_i}\times \underset{\sim}{B})$ $\qquad$ $m_e n_e \left[\frac{\partial U_e}{\partial t} + \underset{\sim}{U_e}\cdot\nabla \underset{\sim}{U_e}\right] = \nabla p_e + q_e n_e(\underset{\sim}{E} + \underset{\sim}{U_e}\times\underset{\sim}{B})$

Eq. of State: $(\gamma_e = 1)$ $\quad p_i = T_i = 0$ $\qquad$ $p_e = n_e T_e$

Poisson's Eq: $\nabla\cdot\underset{\sim}{E} = \frac{\rho_q}{\epsilon_0}$ $\qquad$ $\rho_q = \sum_s n_s q_s$

Faraday's Law: $\nabla\times\underset{\sim}{E} = -\frac{\partial B}{\partial t}$ $\qquad$ $j = \sum_s n_s q_s \underset{\sim}{U_s}$

Ampere/Maxwell Laws: $\nabla\times\underset{\sim}{B} = \mu_0 j + \mu_0\epsilon_0 \frac{\partial E}{\partial t}$

$\nabla\cdot\underset{\sim}{B} = 0$

C. Equilibrium:

1. Ordering: $n_i = n_0^{(x)} + \epsilon n_{i1}$ $\qquad$ $n_e = n_0^{(x)} + \epsilon n_{e1}$

$\underset{\sim}{U_i} = \underset{\sim}{U_{i0}} + \epsilon \underset{\sim}{U_{i1}}$ $\qquad$ $\underset{\sim}{U_e} = \underset{\sim}{U_{e0}} + \epsilon \underset{\sim}{U_{e1}}$

$\underset{\sim}{E} = \cancel{\underset{\sim}{E_0}}^{\,0} + \epsilon \underset{\sim}{E_1}$ $\qquad$ $\underset{\sim}{B} = B_0\hat{z} + \epsilon \underset{\sim}{B_1}$

NOTE: $\frac{\partial}{\partial t} = 0$ for equilibrium "0" quantities. $\qquad p_e = p_{e0}(x) + p_{e1}$

2. Electron Momentum Eq:

a. $\mathcal{O}(1)$: $m_e n_0 \underset{\sim}{U_{e0}}\cdot\nabla \underset{\sim}{U_{e0}} = -\nabla p_{e0} + q_e n_0(\underset{\sim}{U_{e0}}\times \underset{\sim}{B_0})$

b. $-\nabla p_{e0} = -T_e\nabla n_0(x) = -T_e \frac{\partial n_0}{\partial x}\hat{x} = -T_e n_0'\hat{x}$ $\qquad \boxed{n_0' \equiv \frac{\partial n_0}{\partial x}}$

c. For small electron mass, we can neglect LHS, leaving

$$0 = -T_e n_0'\hat{x} + q_e n_0 B_0 \underset{\sim}{U_e}\times\hat{z}$$

By taking $\hat{z}\times$(Eq.), we solve for $\underset{\sim}{U_e}$: $\quad \underset{\sim}{U_{e0}} = \frac{T_e}{q_e B_0}\left(\frac{n_0'}{n_0}\right)\hat{y} + U_{ez0}\hat{z}$

d. Taking $U_{ez0} = 0$, the drift velocity is

Equilibrium Drift Velocity $\longrightarrow$ $\boxed{\underset{\sim}{U_{e0}} = \frac{T_e}{q_e B_0}\left(\frac{n_0'}{n_0}\right)\hat{y}}$

I. C. (Continued)

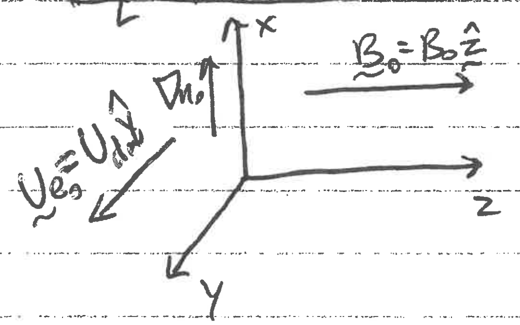
3. DEFINE: Drift Velocity: $\boxed{U_d \equiv \frac{T_e}{q_e B_0}\left(\frac{n_0'}{n_0}\right)}$

4. NOTE: This is just the usual drift due to a general force density $\underset{\sim}{F}$

$$\underset{\sim}{U}_F = \frac{\underset{\sim}{F} \times \underset{\sim}{B}_0}{q n_0 B_0^2}$$ where the force density is electron pressure gradient

$$\underset{\sim}{F} = -\nabla p_{e0}$$

5. Equilibrium Picture:



This is a perfectly good equilibrium that maintains a steady-state drift in the $\hat{y}$-direction.

(Stable with gravity when $\nabla n_0 < 0$
or for any ∇n when force density
$\underset{\sim}{F}_g = mng \ll |-\nabla p_e|$)

6. NOTE: Since $T_i = 0$, $\boxed{\underset{\sim}{U}_{i0} = 0}$. Ions do not drift (no pressure force).

D. Low Frequency Wave Solutions

1. We know for Alfven Waves in Uniform Plasma, $\omega = \pm k_\parallel V_A$.

a. We want to solve for Low Frequency dynamics

$$\boxed{\omega \ll k_\parallel V_A}$$

b. In this limit, the magnetic field is not perturbed, $\underset{\sim}{B}_1 = 0$.

Faraday's Law: $\frac{\partial \underset{\sim}{B}}{\partial t} = -\nabla \times \underset{\sim}{E} \Rightarrow \underbrace{\omega \underset{\sim}{B}_1}_{\text{Small} \to 0} = \underset{\sim}{k} \times \underset{\sim}{E}_1 \Rightarrow \underset{\sim}{k} \times \underset{\sim}{E}_1 = 0 \Rightarrow \boxed{\text{Electrostatic}}$

c. For Electrostatic Perturbations, we may take

$$\boxed{\underset{\sim}{E} = -\nabla \phi}$$

2. NOTE: We'll also assume $\boxed{\omega \ll \omega_{ci}}$ Low Frequency compared to ion cyclotron freq.

Lecture #7 (Continued)

I. D. (Continued)

3. Boltzmann Distribution for Electrons

a. Electron Momentum:

$\mathcal{O}(\epsilon)$: $\cancel{m_e} n_0 [\cancel{\underset{\sim}{U}_{e0}} \cdot \nabla \underset{\sim}{U}_{e1} + \underset{\sim}{U}_{e1} \cdot \nabla \underset{\sim}{U}_{e0}] = -\nabla p_{e1} - q_e n_0 \nabla \phi_1 + q_e n_e (\cancel{\underset{\sim}{U}_{e0} \times \underset{\sim}{B}_1} + \underset{\sim}{U}_{e1} \times \underset{\sim}{B}_0)$

b. For electrostatic perturbation, $\underset{\sim}{B}_1 = 0$.

c. Again, we treat the electron mass as very small $\Rightarrow$ LHS $=0$

d. Thus, we find $0 = -T_e \nabla n_{e1} + q_e n_e (\underset{\sim}{U}_{e1} \times B_0 \hat{\underset{\sim}{z}}) - q_e n_0 \nabla \phi_1$

e. Taking dot product with $\hat{\underset{\sim}{z}}$: $\hat{z} \cdot (\underset{\sim}{U}_{e1} \times \hat{\underset{\sim}{z}}) = 0$, so

$$T_e \frac{\partial n_{e1}}{\partial z} = -q_e n_0 \frac{\partial \phi_1}{\partial z}$$

f. Integrating over z: $\int \frac{1}{n_0} \frac{\partial n_{e1}}{\partial z} dz = \int \frac{-q_e}{T_e} \frac{\partial \phi_1}{\partial z} dz \Rightarrow \ln n_e = \frac{-q_e \phi_1}{T_e} + \text{const.}$

$\Rightarrow \boxed{n_e = n_0 e^{\frac{-q_e \phi_1}{T_e}}}$ Boltzmann Distribution.

Linearized: $e^{\frac{-q_e \phi_1}{T_e}} \simeq 1 - \frac{q_e \phi_1}{T_e} \Rightarrow n_e = n_0 \left(1 - \frac{q_e \phi_1}{T_e}\right)$

$\Rightarrow \boxed{n_{e1} = -n_0 \frac{q_e \phi_1}{T_e}}$

g. Physically, the very low mass electrons move along field line much more rapidly than the wave, thermalizing and giving a Boltzmann distribution. Thus, isothermal approximation $T_e =$ const is consistent.

4. SIMPLIFICATION: Take "1" $\sim e^{i(k_y y + k_\parallel z - \omega t)}$ $\Rightarrow \underset{\sim}{k} \cdot \hat{x} = 0$, ($\underset{\sim}{k}$ in y-z plane)

5. Ion Momentum Equation: (Remember $\underset{\sim}{U}_{i0} = 0$)

a. $\mathcal{O}(\epsilon)$: $m_i \cancel{n_0} \frac{\partial \underset{\sim}{U}_{i1}}{\partial t} = -q_i \cancel{n_0} \nabla \phi_1 + q_i \cancel{n_0} \underset{\sim}{U}_{i1} \times (B_0 \hat{\underset{\sim}{z}})$

b. $-i\omega \underset{\sim}{U}_{i1} = \frac{-q_i}{m_i} i \underset{\sim}{k} \phi_1 + \underbrace{\frac{q_i B_0}{m_i}}_{\omega_{ci}} \underset{\sim}{U}_{i1} \times \hat{\underset{\sim}{z}}$

Lecture #7 (Continued)

I. D.S. (Continued)

c. $\omega \underset{\sim}{U}_{i1} = +\frac{q_i}{m_i} \underset{\sim}{k} \phi_1 + i\omega_{ci} \underset{\sim}{U}_{i1} \times \hat{\underset{\sim}{z}}$

d. Solving for $\underset{\sim}{U}_{i1}$ in terms of ϕ_1:

$$\left.\begin{aligned} \omega U_{ix} &= i\omega_{ci} U_{iy} \\ \omega U_{iy} &= \frac{q_i}{m_i} k_y \phi_1 - i\omega_{ci} U_{ix} \\ \omega U_{iz} &= \frac{q_i}{m_i} k_{\parallel} \phi_1 \end{aligned}\right\} \quad \begin{aligned} U_{ix} &= \frac{i\omega_{ci} \frac{q_i}{m_i} k_y \phi_1}{(\omega^2 - \omega_{ci}^2)} \\ U_{iy} &= \frac{\omega \frac{q_i}{m_i} k_y \phi_1}{(\omega^2 - \omega_{ci}^2)} \\ U_{iz} &= \frac{q_i k_{\parallel}}{\omega m_i} \phi_1 \end{aligned}$$

6. <u>Ion Continuity</u>:

a. $\mathcal{O}(\epsilon)$: $\frac{\partial n_{i1}}{\partial t} + \underset{\sim}{U}_{i1} \cdot \nabla n_0 + n_0 \nabla \cdot \underset{\sim}{U}_{i1} = 0$

b. $$\frac{n_{i1}}{n_0} = -i \frac{n_0'}{n_0} \frac{U_{ix}}{\omega} + \frac{k_y U_{iy}}{\omega} + \frac{k_{\parallel} U_{iz}}{\omega}$$

c. Substituting in for $\underset{\sim}{U}_i$ to yield n_{i1} in terms of ϕ_1:

$$\frac{n_{i1}}{n_0} = \underbrace{\frac{\omega_{ci} \frac{q_i}{m_i} k_y}{\omega(\omega^2 - \omega_{ci}^2)} \left(\frac{n_0'}{n_0}\right) \phi_1}_{①} + \underbrace{\frac{\frac{q_i}{m_i} k_y^2}{(\omega^2 - \omega_{ci}^2)} \phi_1}_{②} + \underbrace{\frac{\frac{q_i}{m_i} k_{\parallel}^2}{\omega^2} \phi_1}_{③}$$

d. In the limit $\omega \ll \omega_{ci}$ (low frequency), we may drop ② and we obtain:

$$\frac{n_{i1}}{n_0} = -\frac{k_y}{\omega B_0}\left(\frac{n_0'}{n_0}\right)\phi_1 + \frac{q_i k_{\parallel}^2}{m_i \omega^2}\phi_1 = \left[\frac{k_y}{\omega} \underbrace{\left\{\frac{T_e}{q_e B_0}\left(\frac{n_0'}{n_0}\right)\right\}}_{U_d} + \frac{\overset{1}{q_i}}{(-q_e)} \frac{T_e}{m_i} \frac{k_{\parallel}^2}{\omega^2}\right]\left(\frac{-q_e \phi_1}{T_e}\right)$$

e. Thus, $$\frac{n_{i1}}{n_0} = \left(\frac{k_y U_d}{\omega} + \frac{k_{\parallel}^2 C_i^2}{\omega^2}\right)\left(\frac{q_e \phi_1}{T_e}\right)$$

where we recall the <u>Ion Acoustic Speed</u> $\boxed{C_i^2 \equiv \frac{T_e}{m_i}}$

From PHYS:4731 Lect #24 (II.F.1.c). Again, we absorb Boltzmann's constant k into temperature T_e to give temperature in energy units. (to avoid confusion with the wavenumber k)

I. D. (Continued)

7. Now we have $n_{e1} = f(\phi_1)$, $n_{i1} = f(\phi_1)$, so we can use Poisson's Equation to solve for linear dispersion relation.

a. $\nabla \cdot \underset{\sim}{E} = \frac{\rho_q}{\epsilon_0} = \frac{n_i q_i + n_e q_e}{\epsilon_0}$ Note: $n_0 q_i + n_0 q_e = 0$ (Neutral Equilibrium)

b. Using $\underset{\sim}{E} = -\nabla\phi$ and linearizing:

$$O(\epsilon): \quad -\nabla^2 \phi_1 = \frac{n_{i1} q_i + n_{e1} q_e}{\epsilon_0} \Rightarrow k^2 \phi_1 = \frac{n_{i1} q_i + n_{e1} q_e}{\epsilon_0}$$

c. $$k^2 \phi_1 = \frac{n_0 q_i^2}{\epsilon_0 T_e}\left(\frac{k_y U_d}{\omega} + \frac{k_\parallel^2 C_i^2}{\omega^2}\right)\phi_1 - \frac{n_0 q_e^2}{\epsilon_0 T_e}\phi_1$$

d. Multiplying by $\frac{T_e}{m_i}$ yields:

$$k^2 C_i^2 = \omega_{pi}^2 \left(\frac{k_y U_d}{\omega} + \frac{k_\parallel^2 C_i^2}{\omega^2}\right) - \omega_{pi}^2$$

e. Eventually, we obtain:

$$\boxed{1 - \frac{\omega_{pi}^2}{k^2 C_i^2}\left(\frac{\omega^2 - \omega k_y U_d - k_\parallel^2 C_i^2}{\omega^2}\right) = 0}$$

Electron Drift Wave Dispersion Relation (Low Frequency Limit)

E. Long Wavelength Drift Waves

1. For long wavelengths $k^2 C_i^2 \ll \omega_{pi}^2$, the dispersion relation simplifies

$$\boxed{\omega^2 - \omega k_y U_d - k_\parallel^2 C_i^2 = 0}$$

2. Solution:

$$\omega = k_y U_d \left[\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{4 k_\parallel^2 C_i^2}{k_y^2 U_d^2}}\right]$$

3. Limits:

a. $U_d \to 0$ ($\nabla n_0 = 0$) $\omega^2 = k_\parallel^2 C_i^2$ Ion Acoustic Waves (Lect # 24)

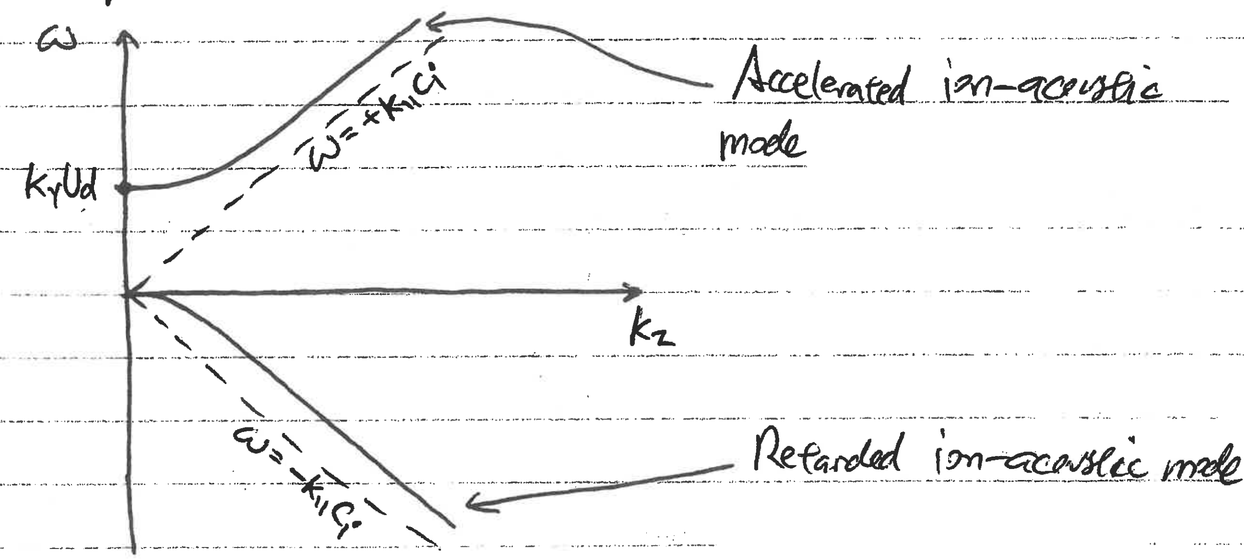
b. $k_\parallel \to 0$ 1. $\omega = k_y U_d$ Drifting Plasma Oscillations

2. $\omega = \frac{-k_\parallel^2 C_i^2}{k_y U_d} \simeq 0$

Lecture #7 (Continued) Howes ⑦

I. E. (Continued)

4. Dispersion Relation: Fixed k_y, ω vs. $k_\parallel$



F. Physics of Drift Waves

1. a. For a uniform plasma, motions with $\nabla \cdot \underset{\sim}{U}_1 = 0$ do not perturb the density $\left(\frac{\partial n_1}{\partial t} = -n_0 \nabla \cdot \underset{\sim}{U}_1 = 0\right)$.

b. But, when a density gradient is present,

$$\frac{\partial n_1}{\partial t} = -U_x \frac{\partial n_0}{\partial x} \neq 0 \text{ even when } \nabla \cdot \underset{\sim}{U}_1 = 0.$$

c. Here, the $\underset{\sim}{E} \times \underset{\sim}{B}$ drift pushes plasma of lower density into higher density regions (and vice versa).

d. For the system we evaluated, this is due to E_y. Since $\underset{\sim}{E}_1 = -\nabla \phi_1$, it is $-ik_y\phi_1$ component that leads to these motions. Thus $k_y \neq 0$ is necessary, otherwise we just have the usual ion-acoustic waves along the magnetic field.

Lecture #7 (Continued)

I.F. (Continued)

2. <u>Low Frequency Turbulence in Fusion Devices</u>

a. Fusion devices (tokamaks, for example) typically have $\beta \sim 1\% \ll 1$.

b. Thus, Alfven waves travel very fast along the mean field.

c. The low frequency turbulent dynamics in tokamaks is <u>Drift wave turbulence</u> due to density gradients in the plasma.

d. Many studies of turbulence in fusion devices employ the <u>electrostatic approximation</u> as outlined here.

3. <u>DEFINE:</u> Drift Wave Frequency $\boxed{\omega_* \equiv k_y U_d}$

a. $$\omega_* = \frac{T_e}{q_e B_0} k_y \left(\frac{n_0'}{n_0}\right)$$

b. $$\omega^2 - \omega\omega_* - k_\parallel^2 c_i^2 = 0$$

c. $$\omega = \omega_* \left(\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{4k_\parallel^2 c_i^2}{\omega_*^2}}\right)$$