Lecture #8 MHD Stability

I. Introduction

A. Confinement: 1. One of the goals of the fusion program in particular, and plasma physics in general, is the confinement of plasmas.

2. We have discussed last semester a number of MHD Equilibria
⇒ BUT NOT ALL MHD EQUILIBRIA ARE STABLE!

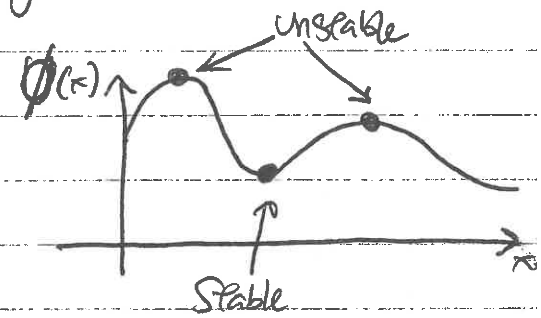
3. Instabilities: Macroscopic vs. microscopic

a. Plasma instabilities that involve the displacement of the plasma are macroscopic, and may be discussed in terms of MHD.

b. Microscopic instabilities, on the other hand, are driven by the structure in the velocity distribution function and require a kinetic description.

4. Unfortunately for the fusion program, many perfectly good MHD equilibria are unstable to a wide range of instabilities, with odd names like interchange, sausage, and kink instabilities.

B. Stability: Simple 1-D Example



1. Consider the problem of a frictionless ball on a surface of varying height

a. Gravitational potential $\phi(x) = mg\,h(x)$

2. The force $F = -\frac{\partial \phi}{\partial x} = 0$ at equilibrium points.

a. But, if we displace the mass by small amount $y = x - x_0$ from the equilibrium point x_0, we have
⇒ Stability if the force causes the particle to return to x_0
⇒ Instability if the force pushes the particle farther from x_0

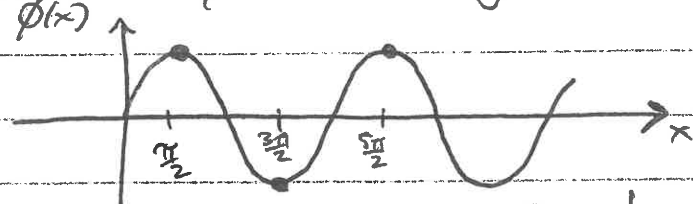
3. The Energy Principle is a powerful energy argument that can easily tell us about stability.

Lecture #8 (Continued)

I. B. (Continued)

4. Consider a system with a gravitational potential $\phi(x) = \phi_0 \sin x$

(We assume $\phi_0 > 0$)



a. Energy of the system is $E = \frac{1}{2} m v^2 + \phi(x)$

b. Let's Taylor expand $\phi(x)$ about an equilibrium point x_0

(Remember $\left.\frac{\partial \phi}{\partial x}\right|_{x_0} = 0$ at equilib. $\Rightarrow \phi_0 \cos x_0 = 0 \Rightarrow x_0 = \left(\frac{2n+1}{2}\right)\pi \quad n = 0,1,2,\ldots$

$$\phi(x) = \phi(x_0) + (x - x_0)\left.\frac{\partial \phi}{\partial x}\right|_{x_0} + \frac{(x-x_0)^2}{2}\left.\frac{\partial^2 \phi}{\partial x^2}\right|_{x_0} + \ldots$$

c. Let's write the small displacement $y = x - x_0$ (also $\frac{dx}{dt} = \frac{dy}{dt}$).

d. Thus,

$$E = \frac{1}{2} m \left(\frac{dy}{dt}\right)^2 + \phi(x_0) + y\left.\frac{\partial \phi}{\partial x}\right|_{x_0} + \frac{y^2}{2}\left.\frac{\partial^2 \phi}{\partial x^2}\right|_{x_0} + \ldots$$

5. Now, for small displacement $y \ll 1$, let's expand in orders:

a. $O(y^0)$ $\quad E_0 = \phi(x_0)$

b. $O(y^1)$ $\quad E_1 = y\left.\frac{\partial \phi}{\partial x}\right|_{x_0}$ (But $\left.\frac{\partial \phi}{\partial x}\right|_{x_0} = 0 \Rightarrow E_1 = 0$)

c. $O(y^2)$ $\quad E_2 = \underbrace{\frac{1}{2} m \left(\frac{dy}{dt}\right)^2}_{\text{Kinetic Energy}} + \underbrace{\frac{y^2}{2}\left.\frac{\partial^2 \phi}{\partial x^2}\right|_{x_0}}_{\text{Potential Energy, } \delta W}$

6. Since we have conservation of energy, as y changes, $E_2 =$ constant.

a. $\boxed{\text{Thus, the particle can \underline{only} gain kinetic energy if } \delta W < 0.}$

b. $\delta W = \underbrace{\left(\frac{y^2}{2}\right)}_{\text{positive definite}} \underbrace{\left(\left.\frac{\partial^2 \phi}{\partial x^2}\right|_{x_0}\right)}_{\text{Depends on sign.}}$

$\frac{\partial^2 \phi}{\partial x^2} = -\phi_0 \sin x$

$\Rightarrow x_0 = \frac{\pi}{2} \quad \frac{\partial^2 \phi}{\partial x^2} = -\phi_0 < 0$ unstable!

$x_0 = \frac{3\pi}{2} \quad \frac{\partial^2 \phi}{\partial x^2} = +\phi_0 > 0$ Stable!

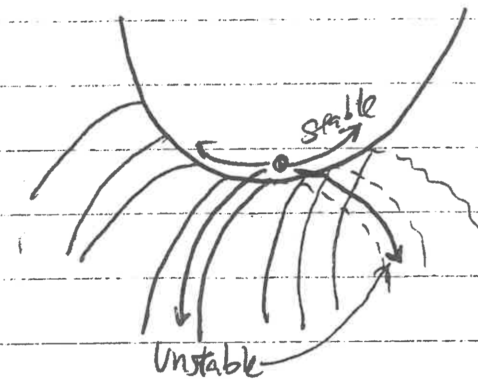
Lecture #8 (Continued)

I. B. (Continued)

7. This example is trivial, but it demonstrates a powerful technique.

a. For more complicated situations, it can easily demonstrate instability.

Ex: 2-D : Saddle Point



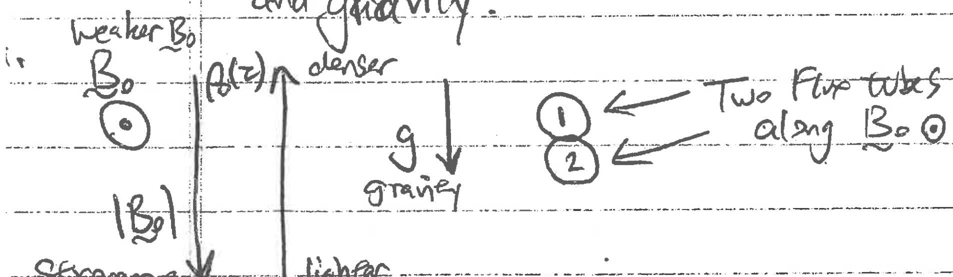
b. In a complicated MHD Equilibrium, the equilibrium geometry can be used to calculate stability (numerically if needed).

II. Types of MHD Instabilities

Momentum Eq: $\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{\mu_0} - \rho\nabla\phi$ ← gravity

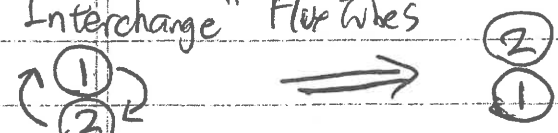
A. Interchange Instability

1. Consider a plasma equilibrium with total force $\nabla\left[p(z) + \frac{[B(z)]^2}{2\mu_0}\right] + \rho\nabla\phi = 0$, but with $\rho(z)$ increasing with height, $B(z)$ decreasing with height, and gravity.



$\rho_1 > \rho_2$
$B_1 < B_2$

b. "Interchange" Flux Tubes



without changing flux tube Volume (Thus magnetic & thermal pressure don't change)

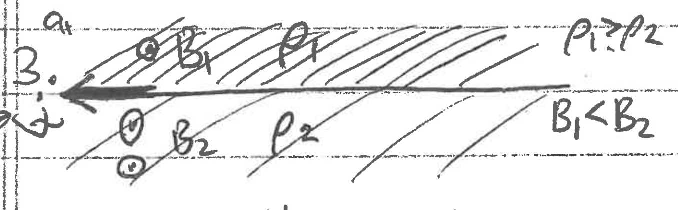
c. Now heavier flux tube 1 has dropped in gravitational field, releasing energy! ⇒ UNSTABLE $\delta W < 0$.

Lecture #8 (Continued)

II. A. (Continued)

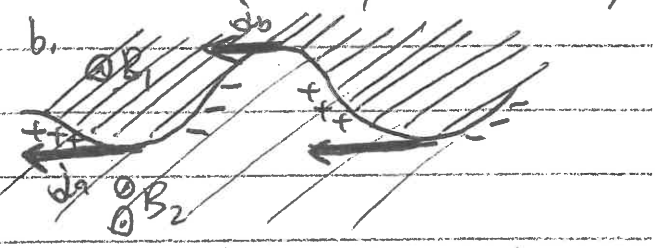
2. a. Field lines are not bent during interchange motion (Magnetic tension does not act as restoring force)
   b. Release of gravitational P.E. drives kinetic energy ⇒ Instability

3. a.



Current along Boundary!

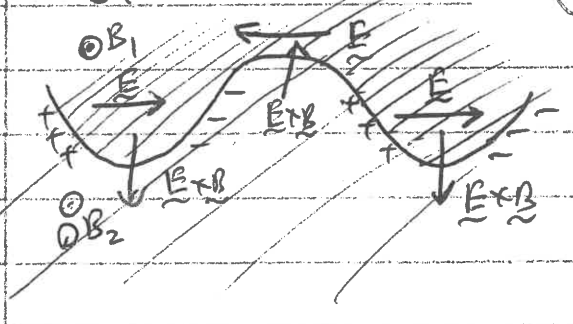
b.



This is the Magnetized Rayleigh-Taylor Instability

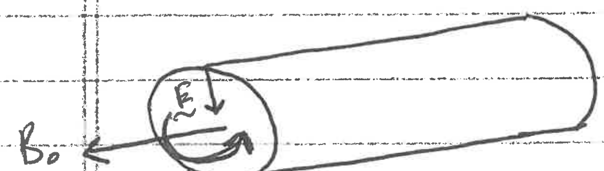
c. More fluid supported by j_a than j_b ⇒ $|j_a| > |j_b|$

d. $\frac{\partial \rho_q}{\partial t} + \nabla \cdot j = 0$ ⇒ Charge build up ⇒ leads to $\underset{\sim}{E}$ fields.

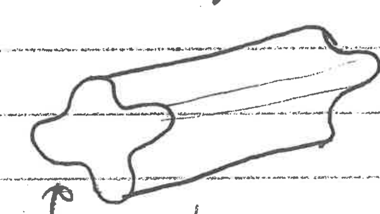


e. $\underset{\sim}{E} \times \underset{\sim}{B}$ velocity reinforces original perturbation.

4. Interchange does not require gravity. Centrifugal force can act like gravity to drive interchange.



$\underset{\sim}{E} \times \underset{\sim}{B}$ flow leads to rotation of plasma

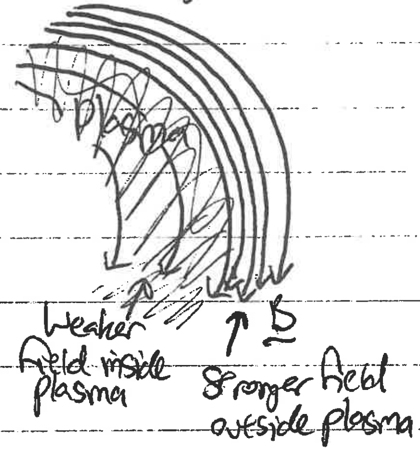


Interchange at plasma edge leads to a "fluted" apperance
⇒ "Flute" Instability

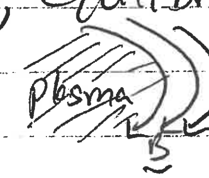
II.

B. Interchange due to Field Curvature

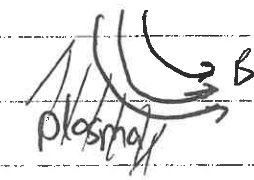
i.



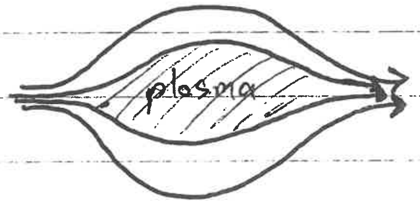
a. When field lines are concave towards the plasma, equilibrium is unstable.



b. When field lines are convex towards the plasma, equilibrium is stable.

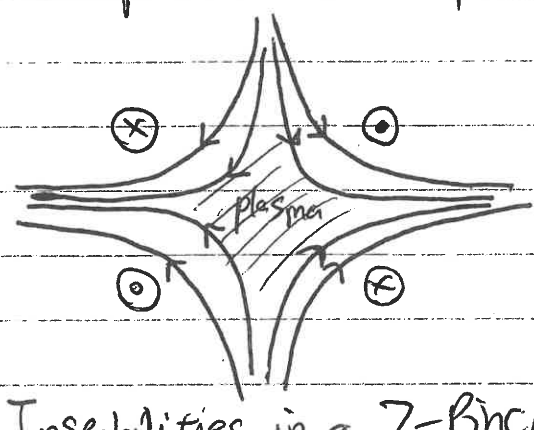


2. Mirror Machine

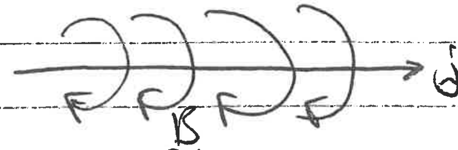


Unstable

3. Cusp Mirror Geometry

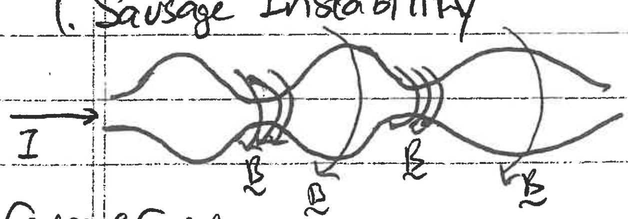


Stable



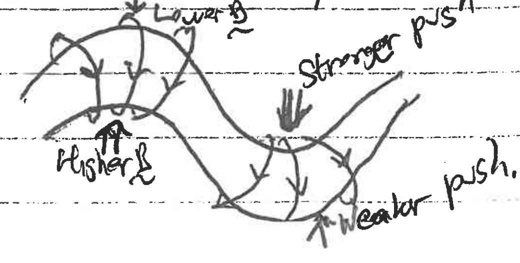
C. Instabilities in a Z-Pinch

1. Sausage Instability



a. Constant Current I along z $\Rightarrow$ Stronger I at small radius $\Rightarrow$ Strong $\underset{\sim}{B}$

2. Kink Instability



III. The Linear Force Operator

A. 1. We want to express the change in the potential energy due to a displacement.

2. Ideal MHD Equations (from last semester)

Continuity: $\frac{\partial \rho}{\partial t} + \underset{\sim}{U} \cdot \nabla \rho = -\rho \nabla \cdot \underset{\sim}{U}$

Momentum: $\rho \frac{\partial \underset{\sim}{U}}{\partial t} + \rho \underset{\sim}{U} \cdot \nabla \underset{\sim}{U} = -\nabla p + \frac{(\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B}}{\mu_0}$

Induction: $\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B})$

Pressure: $\frac{\partial p}{\partial t} + \underset{\sim}{U} \cdot \nabla p = -\gamma p \nabla \cdot \underset{\sim}{U}$

Conserved Energy: $E = \int d^3\underset{\sim}{x} \left[\underbrace{\frac{1}{2}\rho U^2}_{\text{Kinetic Energy}} + \underbrace{\frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0}}_{\text{Potential Energy}} \right]$

3. Linearize Equations, but do not assume $\nabla \times \underset{\sim}{B}_0 = 0$ or $\nabla p_0 = 0$, since the equilibrium fields are not necessarily straight and uniform!

$$\rho = \rho_0 + \rho_1$$
$$\underset{\sim}{U} = \cancel{\underset{\sim}{U}_0}^{\,0} + \underset{\sim}{U}_1$$
$$\underset{\sim}{B} = \underset{\sim}{B}_0 + \underset{\sim}{B}_1$$
$$p = p_0 + p_1$$

(But, for equilibrium, $\frac{\partial p_0}{\partial t} = 0$, $\frac{\partial \underset{\sim}{B}_0}{\partial t} = 0$)

$$\frac{\partial \rho_1}{\partial t} + \underset{\sim}{U}_1 \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \underset{\sim}{U}_1 = 0$$

$$\rho_0 \frac{\partial \underset{\sim}{U}_1}{\partial t} = -\nabla p_1 + \frac{(\nabla \times \underset{\sim}{B}_0) \times \underset{\sim}{B}_1 + (\nabla \times \underset{\sim}{B}_1) \times \underset{\sim}{B}_0}{\mu_0}$$

$$\frac{\partial \underset{\sim}{B}_1}{\partial t} = \nabla \times (\underset{\sim}{U}_1 \times \underset{\sim}{B}_0)$$

$$\frac{\partial p_1}{\partial t} + \underset{\sim}{U}_1 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \underset{\sim}{U}_1 = 0$$

II. A. (Continued)

4. Put in terms of displacement vector $\underset{\sim}{\xi}_1$

a. $\underset{\sim}{U}_1 = \frac{\partial \underset{\sim}{\xi}_1}{\partial t}$

b. For example: Continuity Eq: $\frac{\partial \rho_1}{\partial t} + \frac{\partial \underset{\sim}{\xi}_1}{\partial t} \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \frac{\partial \underset{\sim}{\xi}_1}{\partial t} = 0$

1. Now, we can integrate this equation over time:

$$\int \frac{\partial \rho_1}{\partial t} dt + \left(\int \frac{\partial \underset{\sim}{\xi}_1}{\partial t} dt\right) \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \left(\int \frac{\partial \underset{\sim}{\xi}_1}{\partial t} dt\right) = 0$$

2. $$\boxed{\rho_1 + \underset{\sim}{\xi}_1 \cdot \nabla \rho_0 + \rho_0 \nabla \cdot \underset{\sim}{\xi}_1 = 0}$$

c. Similarly, for Induction Eq: $\boxed{\underset{\sim}{B}_1 = \nabla \times (\underset{\sim}{\xi}_1 \times \underset{\sim}{B}_0)}$

d. Pressure Eq: $\boxed{p_1 + \underset{\sim}{\xi}_1 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \underset{\sim}{\xi}_1 = 0}$

5. We can now substitute $\underset{\sim}{B}_1$ and p_1 in terms of $\underset{\sim}{\xi}_1, p_0, \underset{\sim}{B}_0$ into the Momentum Equation to find:

a. $$\rho_0 \frac{\partial^2 \underset{\sim}{\xi}_1}{\partial t^2} = \nabla\left[\underset{\sim}{\xi}_1 \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \underset{\sim}{\xi}_1\right] + \frac{(\nabla \times \underset{\sim}{B}_0) \times [\nabla \times (\underset{\sim}{\xi}_1 \times \underset{\sim}{B}_0)]}{\mu_0} + \frac{(\nabla \times [\nabla \times (\underset{\sim}{\xi}_1 \times \underset{\sim}{B}_0)]) \times \underset{\sim}{B}_0}{\mu_0}$$

b. We usually write this in terms of the linear force operator $\underset{\sim}{F}(\underset{\sim}{\xi}_1)$

$$\rho_0 \frac{\partial^2 \underset{\sim}{\xi}_1}{\partial t^2} = \underset{\sim}{F}(\underset{\sim}{\xi}_1)$$

where $\underset{\sim}{F}(\underset{\sim}{\xi}_1)$ is the RHS above

6. The linear force operator $\underset{\sim}{F}(\underset{\sim}{\xi}_1)$ has useful mathematical properties that lead to two powerful approaches: 1) Normal Mode Method
2) Energy Principle