Lecture #9 Normal Mode Analysis and the Energy Principle

I. Properties of the Linear Force Operator

A. Review

1. Last time we derived the Linear Force Operator for small displacements $\underset{\sim}{\xi}_1$

$$\rho_0 \frac{\partial^2 \underset{\sim}{\xi}_1}{\partial t^2} = \underset{\sim}{F}(\underset{\sim}{\xi}_1) \quad \text{where}$$

$$\underset{\sim}{F}(\underset{\sim}{\xi}_1) = \nabla\left[\underset{\sim}{\xi}\cdot\nabla p_0 + \gamma p_0 \nabla\cdot\underset{\sim}{\xi}_1\right] + \frac{(\nabla\times\underset{\sim}{B}_0)\times\left[\nabla\times(\underset{\sim}{\xi}_1\times\underset{\sim}{B}_0)\right]}{\mu_0} + \frac{\left(\nabla\times\left[\nabla\times(\underset{\sim}{\xi}_1\times\underset{\sim}{B}_0)\right]\right)\times\underset{\sim}{B}_0}{\mu_0}$$

2. Also, recall the conserved energy in Ideal MHD:

$$E = \int d^3\underset{\sim}{x}\left[\frac{1}{2}\rho U^2 + \frac{p}{\gamma-1} + \frac{B_0^2}{2\mu_0}\right]$$

B. Expansion of MHD Energy in orders of $\underset{\sim}{\xi}_1$

1. Just as we did for the simple mechanical system in I.B. of Lecture #1, we can split the MHD conserved energy into orders of $\underset{\sim}{\xi}_1$.

$\mathcal{O}(|\underset{\sim}{\xi}_1|^0)$ $$E_0 = \int d^3\underset{\sim}{x}\left[\frac{p_0}{\gamma-1} + \frac{B_0^2}{2\mu_0}\right]$$

NOTE: This is the work $= \underset{\sim}{F}\cdot\Delta\underset{\sim}{x}$

$\mathcal{O}(|\underset{\sim}{\xi}_1|)$ $$E_1 = \int d^3\underset{\sim}{x}\ \underset{\sim}{\xi}_1\cdot\left[\nabla p_0 - \frac{(\nabla\times\underset{\sim}{B}_0)\times\underset{\sim}{B}_0}{\mu_0}\right]$$ (This [ ] = 0 in MHD equilibrium)

$\mathcal{O}(|\underset{\sim}{\xi}_1|^2)$ $$E_2 = \underbrace{\int d^3\underset{\sim}{x}\left[\frac{1}{2}\rho_0\left|\frac{\partial\underset{\sim}{\xi}_1}{\partial t}\right|^2\right]}_{\text{Kinetic Energy}} + \underbrace{\delta W(\underset{\sim}{\xi}_1,\underset{\sim}{\xi}_1)}_{\text{Potential Energy}}$$

We'll derive form of δW soon.

C. Self-Adjoint Property

1. We can differentiate E_2 in time to determine a form of δW:

$$\frac{\partial E_2}{\partial t} = \int d^3\underset{\sim}{x}\ \frac{\partial}{\partial t}\left[\frac{1}{2}\rho_0\left|\frac{\partial\underset{\sim}{\xi}_1}{\partial t}\right|^2\right] + \frac{\partial}{\partial t}\left[\delta W(\underset{\sim}{\xi}_1,\underset{\sim}{\xi}_1)\right]$$

a. NOTE: $\frac{\partial}{\partial t}\left[\frac{1}{2}\rho_0\left|\frac{\partial\underset{\sim}{\xi}_1}{\partial t}\right|^2\right] = \rho_0\frac{\partial\underset{\sim}{\xi}_1}{\partial t}\cdot\frac{\partial^2\underset{\sim}{\xi}_1}{\partial t^2}$

Lecture #9 (Continued)

I. C.1. (Continued)

b. NOTE: $\frac{\partial}{\partial t}\left[\delta W(\underset{\sim}{\xi}_1, \xi_1)\right] = \delta W\left(\frac{\partial \xi_1}{\partial t}, \xi_1\right) + \delta W\left(\xi_1, \frac{\partial \xi_1}{\partial t}\right)$

2. But, Conservation of energy implies $\frac{\partial E_2}{\partial t} = 0$, so

$$\int d^3\underset{\sim}{x} \frac{\partial \underset{\sim}{\xi}_1}{\partial t} \cdot \underset{\sim}{F}(\xi_1) = -\left[\delta W\left(\frac{\partial \xi_1}{\partial t}, \underset{\sim}{\xi}_1\right) + \delta W\left(\xi_1, \frac{\partial \xi_1}{\partial t}\right)\right]$$

where we have used $\rho_0 \frac{\partial^2 \underset{\sim}{\xi}_1}{\partial t^2} = \underset{\sim}{F}(\underset{\sim}{\xi}_1)$

3. This statement must be true at $t=0$ when I can choose $\underset{\sim}{\xi}_1$ and $\frac{\partial \xi_1}{\partial t}$ arbitrarily as initial conditions.

$\Rightarrow$ Therefore, it must be true for any chosen vectors $\underset{\sim}{\xi}_1$ and $\underset{\sim}{\eta}_1 = \left(\frac{\partial \xi_1}{\partial t}\right)$

$$\int d^3\underset{\sim}{x}\ \underset{\sim}{\eta}_1 \cdot \underset{\sim}{F}(\underset{\sim}{\xi}_1) = -\left[\delta W(\underset{\sim}{\eta}_1, \underset{\sim}{\xi}_1) + \delta W(\underset{\sim}{\xi}_1, \underset{\sim}{\eta}_1)\right]$$

4. This statement is clearly symmetric under exchange of $\underset{\sim}{\xi}_1$ and $\underset{\sim}{\eta}_1$, so

$$\boxed{\int d^3\underset{\sim}{x}\ \underset{\sim}{\eta}_1 \cdot \underset{\sim}{F}(\underset{\sim}{\xi}_1) = \int d^3x\ \underset{\sim}{\xi}_1 \cdot \underset{\sim}{F}(\underset{\sim}{\eta}_1)}$$

<u>The Linear Force Operator $\underset{\sim}{F}$ is self-adjoint!</u>

D. <u>Form for $\delta W(\underset{\sim}{\xi}_1, \underset{\sim}{\xi}_1)$</u>

1. The property above suggests the following form for δW:

$$\boxed{\delta W = -\frac{1}{2}\int d^3\underset{\sim}{x}\ \underset{\sim}{\xi} \cdot \underset{\sim}{F}(\underset{\sim}{\xi})}$$

<u>NOTE</u>: From this point on, $\underset{\sim}{\xi}$ is understood to be the linearized displacement, so I drop the subscript "1".

II. Normal Mode Analysis:

A. Representation as a Superposition of Normal Modes

1. An arbitrary mode has displacement $\underset{\sim}{\xi}_n(\underset{\sim}{x}, t)$, which satisfies $\rho_0 \frac{\partial^2 \underset{\sim}{\xi}_n}{\partial t^2} = \underset{\sim}{F}(\underset{\sim}{\xi}_n)$

a. $\underset{\sim}{F}$ is a time-independent linear operator on $\underset{\sim}{\xi}_n(\underset{\sim}{x}, t)$, so we can separate space & time dependent parts

$$\underset{\sim}{\xi}_n(\underset{\sim}{x}, t) = \underset{\sim}{\xi}_n(\underset{\sim}{x})\, e^{-i\omega_n t}$$

where we assume a simple-harmonic form for time dependence.

b. Thus, we find

$$-\rho_0 \omega_n^2 \underset{\sim}{\xi}_n(\underset{\sim}{x}) = \underset{\sim}{F}[\underset{\sim}{\xi}_n(\underset{\sim}{x})]$$

c. The general solution for an arbitrary $\underset{\sim}{\xi}(\underset{\sim}{x}, t)$ is the sum of normal modes

$$\underset{\sim}{\xi}(\underset{\sim}{x}, t) = \sum_n \underset{\sim}{\xi}_n(\underset{\sim}{x})\, e^{-i\omega_n t}$$

B. Properties of Normal Modes

1. Property I: ω_n^2 is always real.

Proof: a. Consider the complex conjugate of the equation of motion

$$-\rho_0 \omega_n^{2*} \underset{\sim}{\xi}_n^*(\underset{\sim}{x}) = \underset{\sim}{F}(\underset{\sim}{\xi}_n^*)$$

b. Dot with $\underset{\sim}{\xi}_n$ and integrate over volume $\int d^3x$

By self-adjoint property.

$$-\rho_0 \omega_n^{2*} \int d^3x\, |\underset{\sim}{\xi}_n|^2 = \int d^3x\, \underset{\sim}{\xi}_n \cdot F(\underset{\sim}{\xi}_n^*) = \int d^3x\, \underset{\sim}{\xi}_n^* \cdot F(\underset{\sim}{\xi}_n)$$

$$= -\rho_0 \omega_n^2 \int d^3x\, |\underset{\sim}{\xi}_n|^2$$

c. For $|\underset{\sim}{\xi}_n^2|$ non-zero, this leads to

$$\boxed{\omega_n^{2*} = \omega_n^2} \Rightarrow \boxed{\omega_n^2 \text{ is always real}}$$

Lecture #9 (Continued)

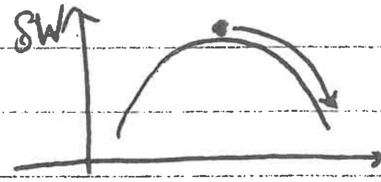
II. B. (Continued)

2. Implications of Property I:

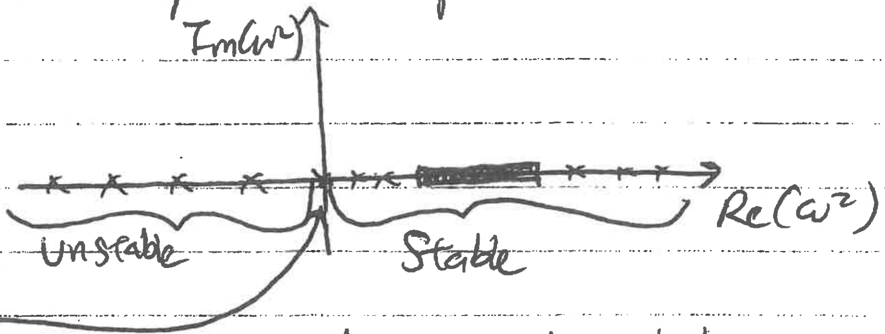
a. For $\omega_n^2 > 0$, ω_n is purely real and eigenfunction is oscillatory. $\Rightarrow$ STABLE

b. For $\omega_n^2 < 0$, ω_n is purely imaginary and eigenfunction undergoes exponential growth due to one root.

$\Rightarrow$ UNSTABLE



c. Numerical Simplification: In solving for roots of the equation of motion (i.e. finding the frequency of the normal mode), one need only look for real ω^2 and need not search all of complex ω^2 space.



d. $\omega_n^2 = 0$ is the point of marginal stability separating stable from unstable solutions.

3. Property II: The eigenmodes of $\underset{\sim}{F}$ are orthonormal,

$$\int d^3\underset{\sim}{x}\, \rho_0\, \xi_m^* \cdot \xi_n = \delta_{mn}.$$

(See Gurnett & Bhattacharjee for proof).
sec 7.3.4

II. (Continued)

C. General Procedure for Solution by Normal Mode Analysis

1. a. The procedure is analogous to the solution of the linear dispersion relation for a given system.
   b. Of course, we take not homogeneous conditions but use the equilibrium solutions for $\rho_0(\underset{\sim}{x})$, $\underset{\sim}{B}_0(\underset{\sim}{x})$
   c. Also, unlike MHD waves in a homogeneous plasma, sources of free energy are present so we may find many unstable eigenmodes (with ω_n purely imaginary).
   d. In fact, stability is more often the exception than the rule.

2. a. Begin with $\rho_0 \frac{\partial^2 \underset{\sim}{\xi}}{\partial t^2} = \underset{\sim}{F}(\underset{\sim}{\xi}) \Rightarrow -\rho_0 \omega_n^2 \underset{\sim}{\xi}_n = \underset{\sim}{F}(\underset{\sim}{\xi}_n)$
   b. For many systems, we can simplify $\underset{\sim}{F}(\underset{\sim}{\xi})$ because some of the terms are zero.
   c. Symmetries of the system can be used to reduce at least some components of the vector operators in $\underset{\sim}{F}(\underset{\sim}{\xi})$ to algebraic operators (For periodicity, we can use a Fourier decomposition)
   d. The vector equation yields a 3×3 matrix equation which can be solved for the eigenfrequencies ω_n.

3. a. This method yields stable frequencies or unstable growth rates for each mode, and can be used to reconstruct the eigenfunctions.
   b. The somewhat complicated normal mode analysis often gives us more information than we need.
   c. Often, we care only if a system is unstable.

   $\Rightarrow$ The Energy Principle is a more easily applied, yet extremely powerful, technique that determines stability.

III. The Energy Principle

A. Necessary and Sufficient Conditions for Stability

1. Instability is relatively easy to prove:
   a. Choose a physically motivated perturbation $\underset{\sim}{\xi}$
   b. Show that this perturbation leads to $\delta W < 0$.
2. Stability is much more difficult to prove
   a. Must show that no perturbation can lead to $\delta W < 0$
   b. Using the energy principle, one may minimize δW with respect to all possible perturbations
   c. If δW_{min} is positive, the system is stable.
3. Remember $E_2 = \underbrace{\int d^3x \left[\frac{1}{2}\rho_0 \left(\frac{\partial \underset{\sim}{\xi}}{\partial t}\right)^2\right]}_{=\delta K} + \delta W(\underset{\sim}{\xi}, \underset{\sim}{\xi}) = \underline{\text{constant}}$

   So $E_2 = \delta K + \delta W$. Note, by definition, $\delta K \geq 0$
4. Theorem I: If $\delta W \geq 0$ for all $\underset{\sim}{\xi}$ then the system is stable.
   $\Rightarrow$ $\delta W \geq 0$ for all $\underset{\sim}{\xi}$ is sufficient for stability

   Proof: a. If $\delta W \geq 0$,

   $$0 \leq \delta K = E_2 - \delta W \leq E_2$$

   b. Thus δK is bounded from above. No unbounded growth of kinetic energy is possible so plasma is stable. QED.
5. Theorem II: If for some function $\underset{\sim}{\eta}$, $\delta W(\underset{\sim}{\eta}, \underset{\sim}{\eta}) < 0$, then the system is unstable.
   $\Rightarrow$ $\delta W \geq 0$ for all $\underset{\sim}{\xi}$ is also necessary for stability.

III. A. 5. (Continued)

Proof: a. Consider a displacement initially such that
$$\underset{\sim}{\xi}(\underset{\sim}{x},0)\neq 0 \quad \text{but} \quad \frac{\partial \underset{\sim}{\xi}}{\partial t}(\underset{\sim}{x},0)=0 \quad \left(\begin{array}{c}\text{displaced but}\\ \text{at rest}\end{array}\right).$$

b. Let this $\underset{\sim}{\xi}$ lead to $\delta W<0$.

c. At time $t=0$, $E_2=\cancel{\delta K}^{\,0}+\delta W<0 \Rightarrow E_2<0$.

d. Define
$$I(t)=\frac{1}{2}\int d^3\underset{\sim}{x}\,\rho_0|\underset{\sim}{\xi}|^2$$

e. Then
$$\frac{d^2I}{dt^2}=\frac{1}{2}\int d^3x\,\rho_0\left[2\left|\frac{\partial\underset{\sim}{\xi}}{\partial t}\right|^2+\underset{\sim}{\xi}^*\cdot\frac{\partial^2\underset{\sim}{\xi}}{\partial t^2}+\underset{\sim}{\xi}\cdot\frac{\partial^2\underset{\sim}{\xi}^*}{\partial t^2}\right)$$
$$=\frac{1}{2}\int d^3\underset{\sim}{x}\left[2\rho_0\left|\frac{\partial\underset{\sim}{\xi}}{\partial t}\right|^2+\underbrace{\underset{\sim}{\xi}^*\cdot\underset{\sim}{F}(\underset{\sim}{\xi})+\underset{\sim}{\xi}\cdot\underset{\sim}{F}(\underset{\sim}{\xi}^*)}_{=2\underset{\sim}{\xi}^*\cdot\underset{\sim}{F}(\underset{\sim}{\xi})=-4\delta W}\right]$$

f. Thus $\frac{d^2I}{dt^2}=2(\delta K-\delta W)$,

but $E_2=\delta K+\delta W$ so $\delta W=E_2-\delta K \Rightarrow \frac{d^2I}{dt^2}=2(2\delta K-E_2)$

g. $\delta K\geq 0$, so let's take $\delta K=0$. Then $\frac{d^2I}{dt^2}=-2E_2>0$
because $E_2<0$.

h. Thus I increases without bound if $\delta W<0$. $\Rightarrow$ UNSTABLE. QED.

6. Thus $\delta W\geq 0$ **for** all $\underset{\sim}{\xi}$ is a <u>necessary</u> and <u>sufficient</u> condition for stability.

<u>B. Forms for δW:</u>

1. We can use $\delta W=-\frac{1}{2}\int d^3\underset{\sim}{x}\,\underset{\sim}{\xi}\cdot \underset{\sim}{F}(\underset{\sim}{\xi})$ and linear force operator $\underset{\sim}{F}(\underset{\sim}{\xi})$ to find useful forms for δW.

III. B. (Continued)

2. After substantial algebra (See Gurnett & Bhattacharjee Sec 7.3.6), we arrive at the form:

$$\delta W = \frac{1}{2}\int d^3x \left[\underbrace{\underbrace{\frac{|\nabla\times(\underset{\sim}{\xi}\times \underset{\sim}{B_0})|^2}{\mu_0}}_{\text{Magnetic Tension and compression}} + \underbrace{\gamma p_0 |\nabla\cdot\underset{\sim}{\xi}|^2}_{\text{Thermal compression}}}_{\text{positive} \Rightarrow \text{stabilizing}} \underbrace{- \underbrace{\underset{\sim}{\xi}^*\cdot \underset{\sim}{j_0}\times[\nabla\times(\underset{\sim}{\xi}\times \underset{\sim}{B_0})]}_{\text{"Kink" Drive}} - \underbrace{\underset{\sim}{\xi}^*\cdot\nabla(\underset{\sim}{\xi}\cdot\nabla p_0)}_{\text{"Interchange" or "Ballooning" Drive}}}_{\text{potentially destabilizing}} \right]$$

3. a. Since thermal compression term is always stabilizing, taking incompressible motions ($\gamma \to \infty$) are always more stable than compressible motions.

b. A fluid with pressure independent of volume ($\gamma \to 0$) is the most unstable.

4. A more complete treatment of a finite volume plasma confined by vacuum magnetic fields includes <u>surface terms</u> and <u>vacuum field energy</u> terms in δW.

C. Application of Energy Principle to Evaluate Stability

1. One may calculate δW for a given equilibrium $p_0(\underset{\sim}{x})$ and $\underset{\sim}{B_0}(\underset{\sim}{x})$ for an arbitrary $\underset{\sim}{\xi}$

2. Eventually, one can minimize δW with respect to each component of $\underset{\sim}{\xi}$. For example, take $\frac{\partial\, \delta W}{\partial \xi_x} = 0$ to find the minimum δW at the solutions for ξ_x minima.

3. Ultimately, one reaches a final value δW_{min}. If $\delta W_{min} < 0$, unstable. Otherwise stable.