PHYS:7730 Homework #5

Reading: (i) Goldreich and Sridhar, ApJ **438**:763 (2006)
Toward a Theory of Interstellar Turbulence. I. Strong Alfvénic Turbulence
(ii) Boldyrev, PRL **96**:115002 (2006)
Spectrum of Magnetohydrodynamic Turbulence

Due at the beginning of class, Thursday, February 24, 2022.

1. **Properties of Incompressible MHD Equations** (cgs units)
The incompressible MHD equations in the symmetrized Elsässer form are given by

$$\frac{\partial \mathbf{z}^{\pm}}{\partial t} \mp \mathbf{v}_A \cdot \nabla \mathbf{z}^{\pm} = -\mathbf{z}^{\mp} \cdot \nabla \mathbf{z}^{\pm} - \nabla P/\rho_0,$$

$$\nabla \cdot \mathbf{z}^{\pm} = 0$$

where $\mathbf{z}^{\pm}(x, y, z, t) = \mathbf{v} \pm \delta\mathbf{B}/\sqrt{4\pi\rho_0}$ are the Elsässer fields given by the sum and difference of the velocity fluctuation $\mathbf{v}$ and the magnetic field fluctuation $\delta\mathbf{B}$ expressed in velocity units.

(a) For a volume integration over all space where you may assume $\mathbf{z}^{\pm} \to 0$ and $p \to 0$ as $\mathbf{r} \to \infty$, show that the energy of one of the Elsasser fields

$$E^{+} = \int d^3\mathbf{r} \rho_0 \frac{|\mathbf{z}^{+}|^2}{2}$$

is a conserved quantity.

(b) Express the conservation of total energy

$$E = \int d^3\mathbf{r} \frac{\rho_0}{2}(|\mathbf{v}|^2 + |\mathbf{b}|^2)$$

and conservation of cross helicity

$$\mathcal{H}_C = \int d^3\mathbf{r} \frac{1}{2}(\mathbf{v} \cdot \mathbf{b})$$

in terms of the Elsässer fields $\mathbf{z}^{\pm}$.
Note that $\mathbf{b} = \delta\mathbf{B}/\sqrt{4\pi\rho_0}$ is the magnetic field perturbation converted to velocity units.

2. **Strong MHD Turbulence in the Inertial Range of Near-Earth Solar Wind Turbulence**
Turbulence in the solar wind can be modeled by strong, isotropic driving with driving amplitude $\delta B_{\perp 0} \simeq B_0$ at an isotropic driving scale $L_0 = L_{\perp 0} = L_{\parallel 0} = 2 \times 10^6$ km. The mean magnetic field has a magnitude $B_0 = 10^{-4}$ G , the ion temperature is typically $T_i \simeq 4.5 \times 10^4$ K, and the ion number density is $n_i = 20$ cm^{-3}. These parameters yield a thermal ion Larmor radius of $\rho_i \simeq 3 \times 10^1$ km. The inertial range of strong MHD turbulence is the range $k_0\rho_i < k_\perp\rho_i < 1$, where the *isotropic driving wavenumber* is given by $k_0 = 2\pi/L_0$ (keep the 2π here).

(a) What is the predicted anisotropy $k_\parallel/k_\perp$ when the cascade reaches the ion scale at $k_\perp\rho_i = 1$ assuming the GS95 model?

(b) What is the predicted anisotropy $k_\parallel/k_\perp$ when the cascade reaches the ion scale at $k_\perp\rho_i = 1$ assuming the B06 model?

(c) What is the predicted anisotropy in the perpendicular plane $k_i/k_\perp$ when the cascade reaches the ion scale at $k_\perp\rho_i = 1$ assuming the B06 model?

(d) Compute the amplitude of the perpendicular magnetic field fluctuations relative to the mean magnetic field $\delta B_\perp / B_0$ at the ion scale at $k_\perp \rho_i = 1$ assuming the GS95 model.

(e) Compute the amplitude of the perpendicular magnetic field fluctuations relative to the mean magnetic field $\delta B_\perp / B_0$ at the ion scale at $k_\perp \rho_i = 1$ assuming the B06 model.

(f) The one-dimensional energy spectrum $E(k_\perp)$ as a function of $k_\perp$, in proper energy density units (meaning we include the factor of $\rho_0/2$), can be integrated over a range of perpendicular wavenumbers $k_{\perp 1} \leq k_\perp \leq k_{\perp 2}$ to yield the energy density ΔE in that range, given by

$$\Delta E = \int_{k_{\perp 1}}^{k_{\perp 2}} dk_\perp E(k_\perp)$$

where

$$E(k_\perp) = \frac{\rho_0}{2} \frac{[v_\perp(k_\perp)]^2}{k_\perp}.$$

Given the details of the strong turbulent cascade in the solar wind above, compute the energy density contained in the GS95 turbulent cascade model over the perpendicular wavenumber ranges
(i) $10^{-4} < k_\perp \rho_i < 10^{-3}$,
(ii) $10^{-1} < k_\perp \rho_i < 1$.
Please provide your answer in cgs units of ergs/cm^3. You may assume that the turbulent fluctuations are Alfvénic.