PHYS:7730 Homework #6

Reading: Howes, *Kinetic Turbulence*, chapter in "Magnetic Fields in Diffuse Media," ed. Lazarian, A., de Gouveia Dal Pino, E. M., and Melioli, C. Springer: New York (2015).

Due at the beginning of class, Thursday, March 3, 2022.

1. **Parallel Wavenumber and Frequency Energy Spectral Scaling for KAW Turbulence**
For a kinetic turbulent cascade over the range of scales $k_\perp \rho_i > 1$, the one-dimensional, perpendicular wavenumber spectrum of the magnetic energy for a kinetic Alfvén wave cascade is predicted to scale as $E_B(k_\perp) \propto k_\perp^{-7/3}$ (see Lec #11, Sec II.A).

   (a) What is the predicted scaling of the one-dimensional, *parallel wavenumber spectrum* of the magnetic energy for a kinetic Alfvén wave cascade, $E_B(k_\parallel)$?

   (b) What is the predicted scaling of the *frequency spectrum* of the magnetic energy for a kinetic Alfvén wave cascade, $E_B(\omega)$?

2. **Kinetic Alfvén Wave Cascade at the Ion Cyclotron Frequency**
As turbulence cascades to smaller length scales, or equivalently to higher wavenumbers, the frequency of the fluctuations also increases with the wavenumber. The Alfvénic turbulent cascade in a weakly collisional plasma can be interrupted by strong ion cyclotron damping when the frequency of the Alfvén and kinetic Alfvén wave cascade reaches the ion cyclotron frequency, $\omega/\Omega_i = 1$. Note that the frequency of the Alfvén and kinetic Alfvén wave can be expressed in terms of the ion cyclotron frequency by

$$\frac{\omega}{\Omega_i} = \left(\frac{\omega}{k_\parallel v_A}\right)\left(\frac{k_\parallel v_{ti}}{\Omega_i}\right)\left(\frac{v_A}{v_{ti}}\right) = \overline{\omega}\frac{k_\parallel \rho_i}{\sqrt{\beta_i}}$$

where $\overline{\omega} \equiv \omega/k_\parallel v_A$.

Consider the kinetic turbulent cascade in the solar wind at 1 AU with the following properties:
(i) strong, isotropic driving with driving amplitude $\delta B_{\perp 0} \simeq B_0$ at an isotropic driving scale $L_0 = L_{\perp 0} = L_{\parallel 0} = 3 \times 10^6$ km;
(ii) mean magnetic field has a magnitude $B_0 = 10^{-4}$ G;
(iii) the ion temperature $T_i \simeq 1.5 \times 10^5$ K and electron temperature $T_e \simeq 1.5 \times 10^5$ K;
(iv) ion number density is $n_i = 20$ cm^{-3};
(v) thermal ion Larmor radius of $\rho_i \simeq 5 \times 10^1$ km.
Define the *isotropic driving wavenumber* by $k_0 = 2\pi/L_0$ (keep the 2π here).

   (a) What is the scaling of the angular frequency as a function of $k_\perp$ of the Alfvén wave and kinetic Alfvén wave turbulent cascade in the MHD regime, $k_\perp \rho_i \ll 1$? What is the scaling of the angular frequency in the KAW regime, $k_\perp \rho_i \gg 1$?

   (b) For a kinetic turbulent cascade model (based on GS95 scalings), compute the normalized perpendicular wavenumber $k_\perp \rho_i$ at which the turbulent fluctuations will reach the ion cyclotron frequency, $\omega/\Omega_i = 1$.
   HINT: For simplicity, you may assume that the MHD scalings for $k_\perp \rho_i \ll 1$ apply for all scales $k_\perp \rho_i < 1$, that the KAW scalings for $k_\perp \rho_i \gg 1$ apply for all scales $k_\perp \rho_i > 1$, and that the two cascades connect continuously at $k_\perp \rho_i = 1$, but with a possible discontinuity in slope.

   (c) Sketch a log-log plot of the frequency ω/Ω_i as function of $k_\perp \rho_i$, indicating the slopes in the regimes $k_\perp \rho_i \ll 1$ and $k_\perp \rho_i \gg 1$ and the position where $\omega/\Omega_i \to 1$.

   (d) In the inner heliosphere, the ion plasma beta β_i is expected to decrease in the strongly magnetized plasma closer to the solar corona. Instead of the conditions above, we take parameters of strongly driven turbulence with $L_0 = L_{\perp 0} = L_{\parallel 0} = 1.5 \times 10^4$ km, $B_0 = 4 \times 10^{-3}$ G, $T_i = T_e \simeq 5 \times 10^5$ K, $n_i = 100$ cm^{-3}, and $\rho_i \simeq 2.4$ km.
   In this case, compute the normalized perpendicular wavenumber $k_\perp \rho_i$ at which the turbulent fluctuations will reach the ion cyclotron frequency, $\omega/\Omega_i = 1$.