PHYS:7730 Homework #7

Suggested Reading: Read GB17 Chapter 8, Sec 8.1–8.3 (p. 281–308)

Due at the beginning of class, Thursday, April 7, 2022.

1. **Hydrodynamic Rankine-Hugoniot Jump Conditions**
   Here we will work through some of the calculations to derive the Rankine-Hugoniot jump conditions for a shock in a hydrodynamic fluid.

   (a) Derive the hydrodynamic Euler equations in conservative form:
   First, we begin with the Euler equations for hydrodynamics, equivalent to taking the ideal MHD equations with $\mathbf{B} = 0$ (see Lecture#6 Sec. I)

$$\frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho = -\rho \nabla \cdot \mathbf{u} \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p \tag{2}$$

$$\frac{\partial p}{\partial t} + \mathbf{u} \cdot \nabla p = -\gamma p \nabla \cdot \mathbf{u} \tag{3}$$

   We want to manipulate these equations into a conservative form, showing that the rate of change of the density of a conserved quantity is equal to the divergence of a flux of that conserved quantity,

$$\frac{\partial q}{\partial t} + \nabla \cdot \mathbf{Q} = 0. \tag{4}$$

   To simplify the algebra (and avoid tensor calculus), we will assume variations only in the direction normal to the shock, $\partial/\partial x \neq 0$ and $\partial/\partial y = \partial/\partial z = 0$. Take the upstream flow to be parallel to the normal $\mathbf{u} = u\hat{\mathbf{x}}$, and we will only solve for the x-component of the momentum density. Derive the equations in this form for the conserved quantity densities q equal to:
   (i) the mass density ρ,
   (ii) the x-component of the momentum density ρu,
   (iii) energy density $\rho u^2/2 + p/(\gamma - 1)$.
   Please box the final form of each of three equations in conservative form.

   (b) In the shock frame of reference (in which the shock front is at rest), we take the upstream quantities to have subscript 1, and the downstream quantities to have subscript 2. Assuming steady-state flow through the shock, use the solutions to part (a) to compute the conservation relations connecting the upstream and downstream fluxes of mass density, 1D momentum density, and energy density. Box the final form of each of the three equations.

   (c) Determining the density jump condition:
   Take the Mach number to be defined by the upstream flow velocity u_1 and the upstream sound speed $c_{s1}^2 = \gamma p_1/\rho_1$, in the form

$$M = \frac{u_1}{c_{s1}}. \tag{5}$$

   Use the relations in (b) to obtain the Rankine-Hugoniot jump condition for the density, giving the ratio of the downstream to the upstream mass density ρ_2/ρ_1 in terms of *only* the adiabatic index γ and the Mach number M.

   (d) For a three-dimensional monatomic gas, what is the numerical value of the density jump in the limit of large Mach number, $M \to \infty$?