PHYS:7730 Homework #8

Suggested Reading: Read Sec 5.1–5.2 (p.149-176) of
Balogh and Treumann, Chapter 5:
Quasi-perpendicular Supercritical Shocks

Due at the beginning of class, Thursday, April 21, 2022.

1. **Parallel MHD Shocks**
Beginning with the equation for the shock compression $r = \rho_2/\rho_1$, given by

$$\begin{aligned} \cos^2\theta_{Bn}(2M_A^2 + 5\beta_1\cos^2\theta_{Bn})r^3 \quad &+ \quad M_A^2\left[M_A^2 - \cos^2\theta_{Bn}(5M_A^2 + 8 + 10\beta_1)\right]r^2 \\ &+ \quad M_A^4(11\cos^2\theta_{Bn} + 2M_A^2 + 5 + 5\beta_1)r - 8M_A^6 = 0 \end{aligned} \tag{1}$$

   (a) Simplify the equation for a parallel shock geometry with $\theta_{Bn} = 0$, presenting the result as a cubic equation in r. Your answer should be in the form $C_3r^3 + C_2r^2 + C_1r + C_0 = 0$.

   (b) Factor this equation so that the three solutions can easily be found. Your result should be the product of three factors that are each linear in r (some of these factors may be identical) set equal to zero.
   Hint: Start by factoring out the expression $(2M_A^2 + 5\beta_1)$.

   (c) Compute the threshold Alfvén Mach number for a shock for each of the solutions in the step above by setting $r = 1$ and solving for M_A in each of the factors from the previous step. Please express these threshold values as $M_A = f(\beta_i)$, where the right-hand side is expressed as a function of β_1 and constants.

   (d) Please give a physical interpretation for each of these shock solutions in this parallel shock limit with $\theta_{Bn} = 0$.
   Hint: Relate each solution to the linear MHD wave modes.

2. **Threshold Inflow Velocity for Oblique Shocks in the $\beta_1 \to 0$ limit**
Here we will solve for the threshold inflow velocity u_1 for oblique MHD shocks in the $\beta_1 \to 0$ limit.

   (a) Beginning with equation (1) in the previous problem, first simplify the notation by taking $x = M_A^2$ and $y = \cos^2\theta_{Bn}$. To determine the threshold for a shock, set $r = 1$ and simplify the equation to a cubic in x of the form $C_3x^3 + C_2x^2 + C_1x + C_0 = 0$. Do not yet take the $\beta_1 \to 0$ limit.

   (b) Although the expression above can be factored in x for arbitrary β_1, such an exercise is a significant challenge. Instead, let's take the $\beta_1 \to 0$ limit. Write down the simplified form of the cubic equation for x in this limit.

   (c) Next, solve for the solutions of x in this equation in terms of y.

   (d) Replace x and y by their original definitions, and interpret the solutions you obtain in terms of the types of shocks to which they correspond. Please write the answers for the thresholds in the form of u_1 equal to an expression which may depend on v_A or θ_{Bn}.