Hawes ①
May 2020

Lecture #11 Kinetic Turbulence & Cascade Models

I. Turbulence in Weakly Collisional Plasmas

A. MHD vs. Kinetic Turbulence

1. MHD is formally valid only in the strongly collisional limit,

$$\underset{\text{collision frequency}}{\nu} \gg \underset{\text{typical wave or fluctuation frequency}}{\omega}$$

2. In MHD, the ion Larmor radius is formally infinitesimal,

$$\underset{\text{Ion Larmor radius, or gyroradius}}{\rho_i} \ll \underset{\text{Typical length scales of interest}}{L}$$

3. But, many space & astrophysical plasmas are weakly collisional ($\nu \ll \omega$), particularly at the scales of the ion gyroradius, ($L \sim \rho_i$)
   a. Gyroradius ρ_i keeps turbulence "fluid" in perpendicular direction!

4. To understand the fate of turbulent energy (how it is dissipated) in space and astrophysical plasmas, we need to move beyond MHD theory and study the <u>kinetic plasma physics of turbulence in weakly collisional plasmas</u>, or

$$\boxed{\text{Kinetic Turbulence}}$$

Lecture #11:

I. (Continued)

B. Turbulence in the Solar Wind

1. The solar wind is a weakly collisional plasma that supports turbulence over a large range of scales.

a. Mean free path in Solar wind: $\lambda_{mfp} \sim 1AU = 1.5\times10^{13}$ cm

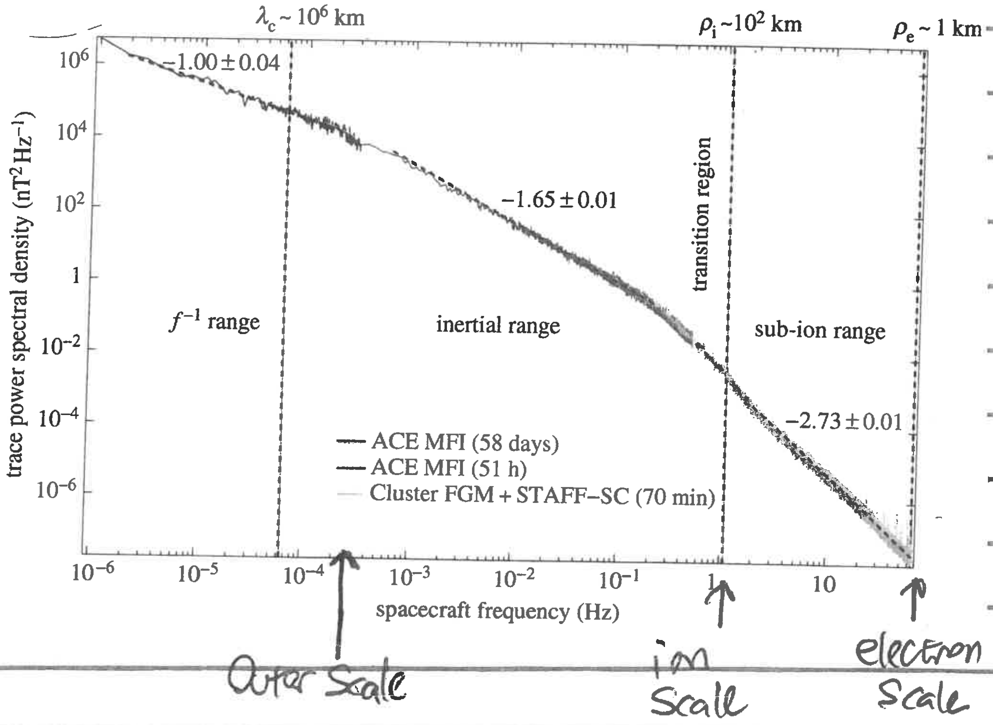
$= 1.5\times10^{8}$ km

$= 150{,}000{,}000$ km

b. Outer Scale of Turbulence: $L \sim 10^{11}$ cm $\sim 10^{6}$ km

2. Observed Frequency Spectrum of Magnetic Energy

Ref: Kiyani, Osman & Chapman, Phil. Trans. Roy. Soc. A. (2015)



a. Unlike MHD turbulent spectrum, the Kinetic turbulence spectrum shows features which may be explained by the properties of kinetic plasma physics.

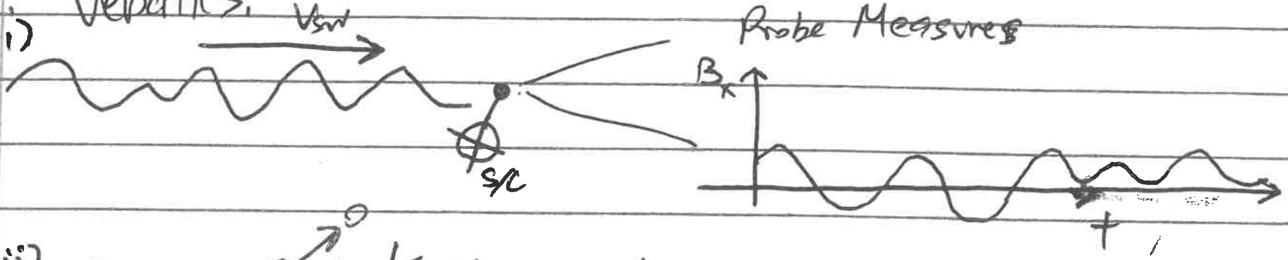
Lecture #11 Howes ③

I. B. (Continued)

3. Theoretical Picture of Turbulent Cascade

a. Taylor Hypothesis: The frequency of fluctuations in the spacecraft frame is dominated by the sweeping of spatial structure past the spacecraft at Super-Alfvenic velocities.

i)



ii) $\omega_{sc} = \cancelto{0}{\omega} + \underset{\sim}{k} \cdot \underset{\sim}{v}_{sw} \simeq \underset{\sim}{k} \cdot \underset{\sim}{v}_{sw}$

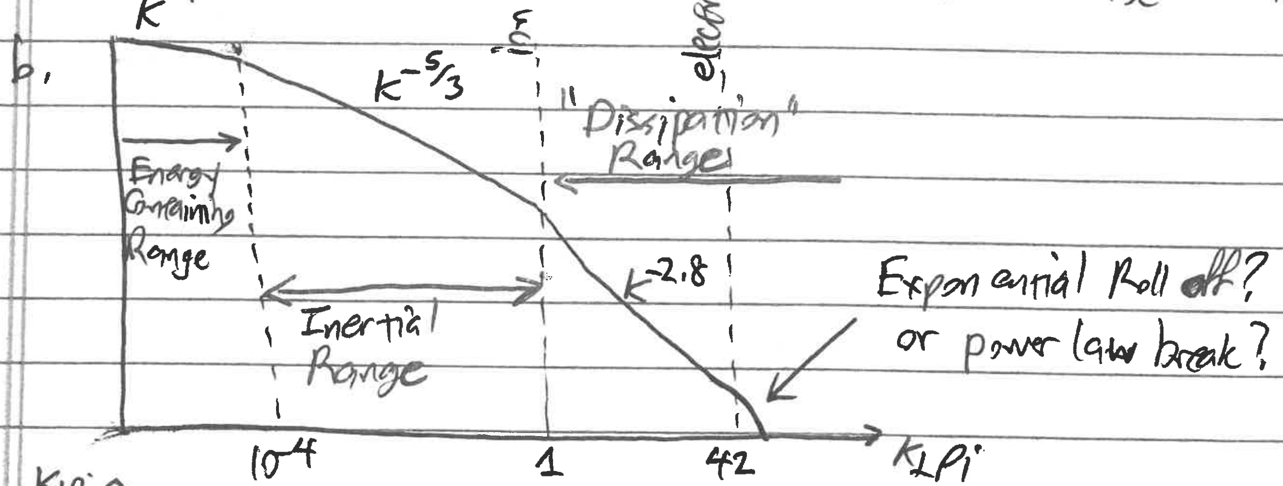
Spacecraft frame frequency

plasma frame frequency

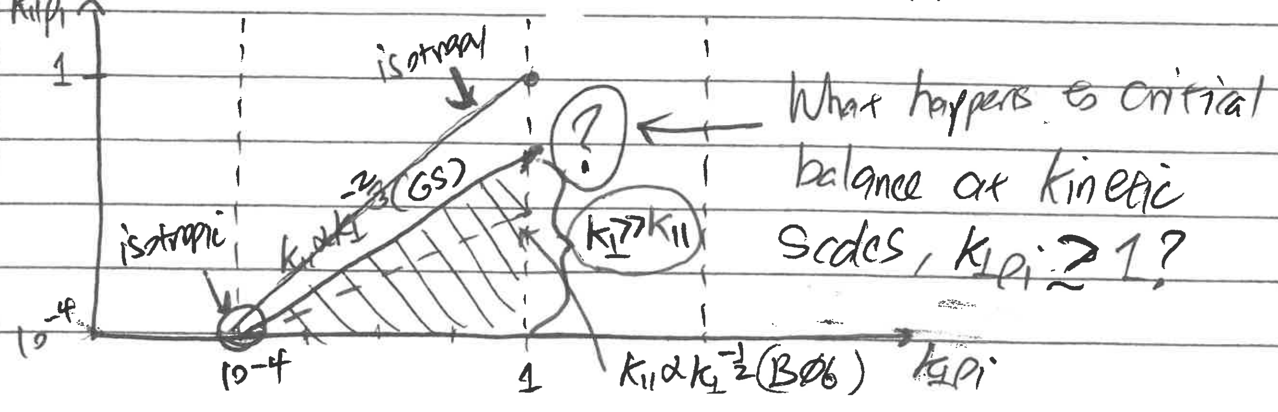
iii) Thus, $\omega_{sc} = 2\pi f_{sc} = k v_{sw} \cos\theta_{kv} \Rightarrow \boxed{k = \frac{2\pi f_{sc}}{v_{sw} \cos\theta_{kv}}}$

Convert from f_{sc} to k!

b.



Anisotropy (Critical Balance)



Lecture #11:

I. B. (Continued)

4. Early Studies speculated on cause of break at ion scales and steepening of spectrum at smaller scales.

a. Location of Break: i) Ion cyclotron frequency $\Omega_i = \frac{q_i B}{m_i c}$ (cgs units)

ii) Ion Larmor Radius, $\rho_i = \frac{v_{ti}}{\Omega_i}$

iii) Ion Inertial Length, $d_i = \frac{v_A}{\Omega_i}$

b. NOTE: i) $\rho_i = \frac{v_{ti}}{\Omega_i} = \left(\frac{v_A}{\Omega_i}\right)\left(\frac{v_{ti}}{v_A}\right) = d_i \beta_i^{\frac{1}{2}} \Rightarrow \boxed{\rho_i = d_i \beta_i^{\frac{1}{2}}}$

ii) Ion Plasma Beta: $\beta_i = \frac{8\pi n_i T_i}{B_0^2} = \frac{v_{ti}^2}{v_A^2}$

c. Steepening of Spectrum: i) Dispersive nature of Kinetic Alfven Waves

ii) Dispersive nature of whistler waves

$\left.\begin{array}{l}\text{iii) Ion cyclotron damping}\\ \text{iv) Landau damping}\end{array}\right]$ Collisionless damping

v) Intermittency (more recent suggestion)

5. A key to understanding the properties of kinetic turbulence is the Kinetic physics of linear waves in a collisionless plasma.

Lecture #1: I. (Continued)

C. Linear Collisionless Dispersion Relation:

1a. For a fully ionized, proton-electron plasma with isotropic Maxwellian equilibrium velocity distributions, the Vlasov-Maxwell dispersion relation gives complex wave frequencies

$$\boxed{\bar{\omega} = \frac{\omega}{k_\parallel v_A}}$$

$$\boxed{\bar{\omega} = \bar{\omega}_{VM}\left(k_\perp \rho_i, \beta_i, T_i/T_e, \frac{k_\parallel}{k_\perp}, \frac{v_{ti}}{c}\right)}$$

b. In the solar wind, $\frac{v_{ti}}{c} \ll 1$ (non-relativistic)

c. In anisotropic turbulence, $k_\parallel / k_\perp \ll 1$

d. In these limits,

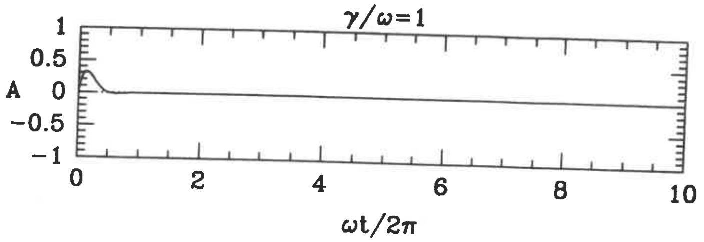
$$\bar{\omega}_{VM}\left(k_\perp \rho_i, \beta_i, \frac{T_i}{T_e}, 0, 0\right)$$

2. Gyrokinetics: Valid for $\frac{v_{ti}}{c} \ll 1$ and $\frac{k_\parallel}{k_\perp} \ll 1$

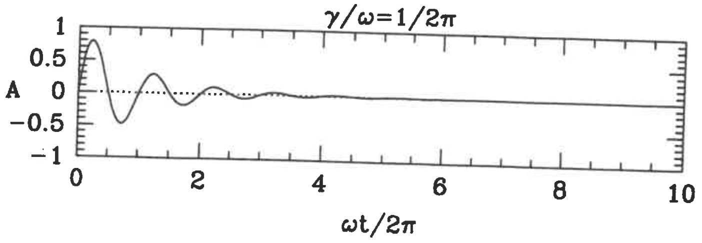
$$\boxed{\bar{\omega} = \bar{\omega}_{GK}\left(k_\perp \rho_i, \beta_i, \frac{T_i}{T_e}\right)}$$

3. Collisionless Damping

a. Complex $\underset{\text{Real frequency}}{\omega} - i \underset{\text{Damping Rate}}{\gamma}$



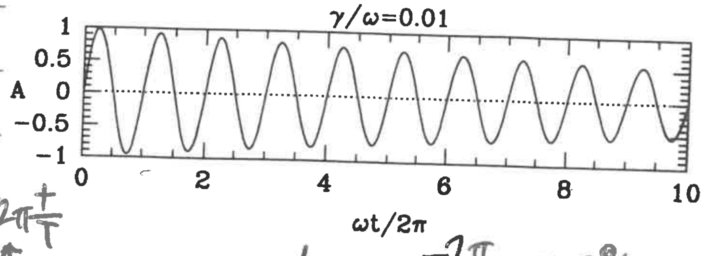
b. Ratio $\frac{\gamma}{\omega}$ gives strength of wave damping



c. When is damping strong?

Rule of thumb

$$\boxed{\frac{\gamma}{\omega} \gtrsim 0.1}$$

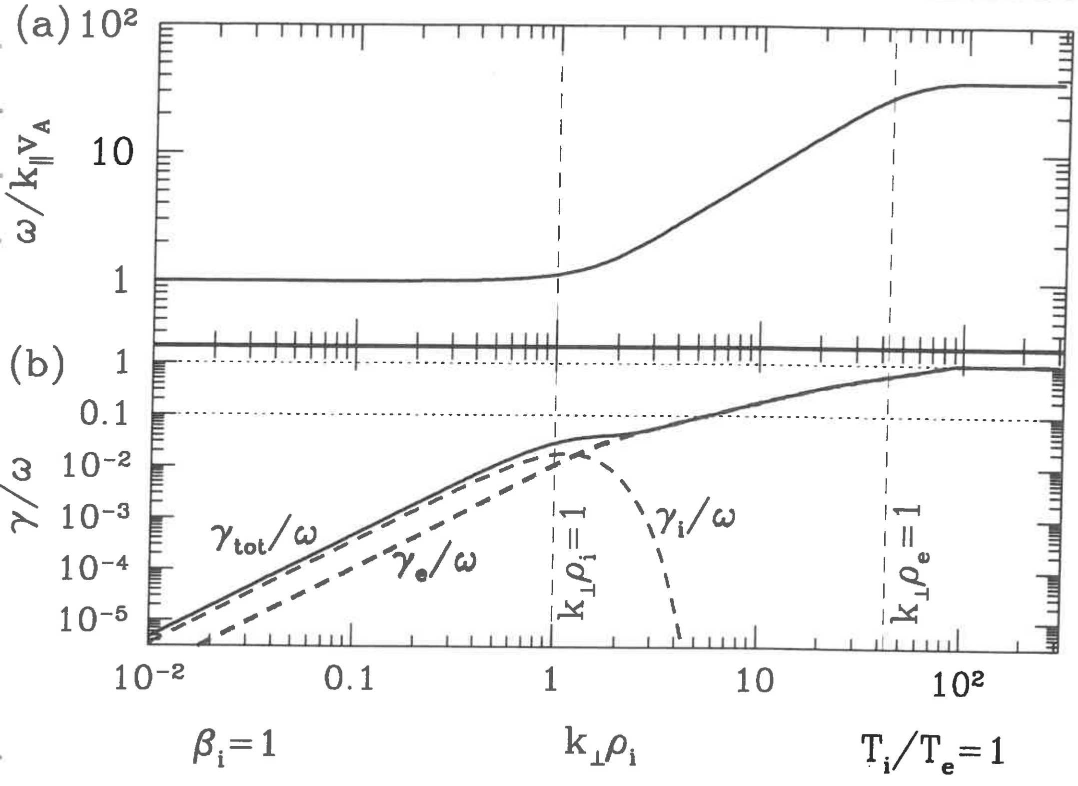


d. $e^{-i(\omega - i\gamma)t} = e^{-i\omega t} e^{-\gamma t} = e^{-i\omega t} e^{-\left(\frac{\gamma}{\omega}\right) 2\pi \frac{t}{T}}$ ← Factor of 2π! $e^{-2\pi} \simeq 0.2\%$

Lecture #11:

I. C. (Continued)

4. VM or GK Dispersion Relation for Alfven Waves



a. MHD Alfven Wave ($k_\perp\rho_i \ll 1$)

$$\omega = k_\parallel v_A$$

b. Kinetic Alfven Wave ($k_\perp\rho_i \gg 1$)

$$\omega = k_\parallel v_A \frac{k_\perp\rho_i}{\sqrt{\beta_i + \frac{2}{(1+T_e/T_i)}}}$$

Alfven/KAW Dispersion Relation

$$\boxed{\omega = k_\parallel v_A \sqrt{1 + \frac{(k_\perp\rho_i)^2}{\beta_i + \frac{2}{(1+T_e/T_i)}}}}$$

Ref: Howes, Klein, & TenBarge ApJ (2014)

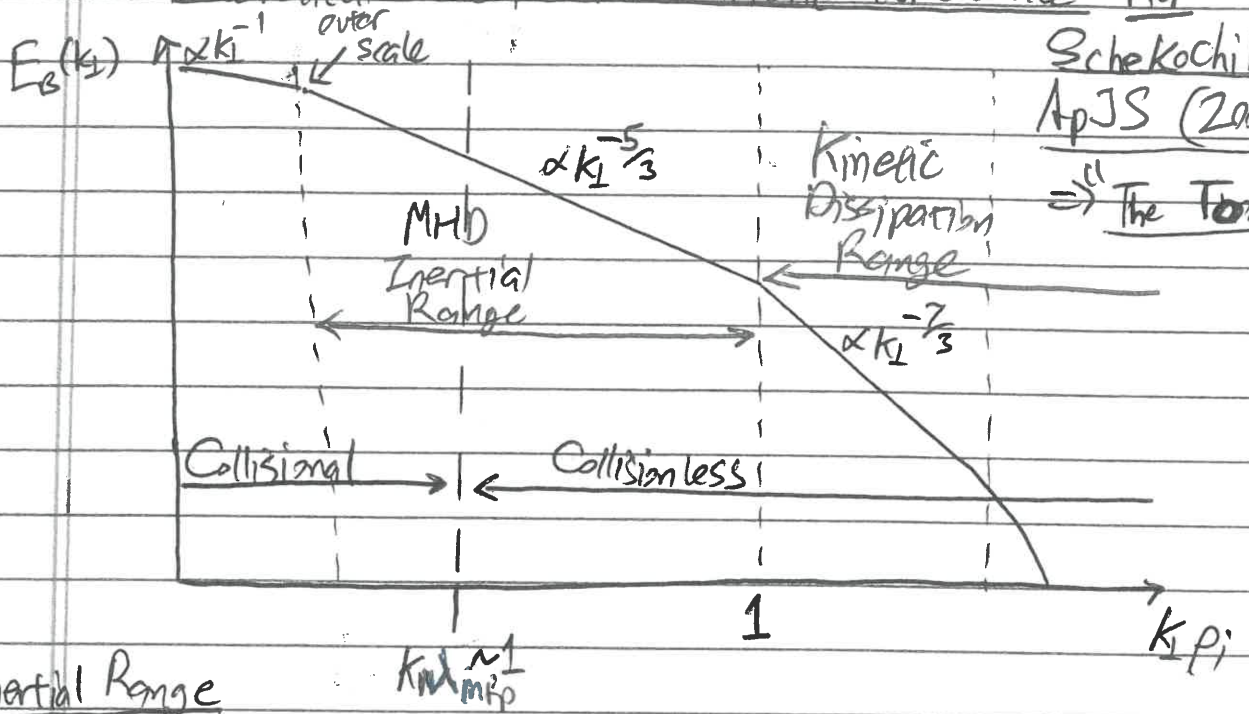
c. At $k_\perp\rho_i \gtrsim 1$, Alfven wave becomes dispersive!

d. Damping:
- i) Ion damping peaks at $k_\perp\rho_i \sim 1$
- ii) Electron damping increases monotonically
- iii) Damping becomes strong at $k_\perp\rho_i \sim 5$
- iv) Damping is extremely strong at $k_\perp\rho_e \sim 1$

Lecture #11 I. (Continued)

D. Theoretical Results for Kinetic Turbulence

Ref: Schekochihin et al. ApJS (2009) ⇒ "The Tome"



Inertial Range

1. In the turbulent inertial range, the scale-dependent anisotropy (due to critical balance) leads to more anisotropy at smaller scales, $\boxed{\frac{k_\parallel}{k_\perp} \ll 1}$

2. In this limit (a) the nonlinear dynamics of the Alfven waves (incompressible fluctuations) are fluid in nature and described by reduced MHD.

(b) The collisionality does not change the Alfvenic dynamics, so the anisotropic Alfvenic cascade simply passes through the collisional to collisionless transition at $k_\parallel \lambda_{mfp} \sim 1$.

(c) The scaling is that predicted by GS95 theory, $E_B \propto k_\perp^{-5/3}$

(d) This cascade continues down to $k_\perp \rho_i \sim 1$.

3. a. Compressible fluctuations (fast & slow magnetosonic waves) undergo a transition at $k_\parallel \rho_i \sim 1$, so we need kinetic theory.

b. Compressible modes may be damped by ion viscosity or collisionless Landau Damping.

Lecture #11 III. D. (Continued)

4. Observationally, there is <u>very little compressible fast wave energy</u> in solar wind turbulence
   <u>Ref:</u> Howes et al. ApJL (2012)

5. The slow wave fluctuations in solar wind turbulence can be shown to be passively advected by the Alfvenic turbulence at $k_\perp \rho_i \ll 1$. <u>Ref:</u> The Tome

6. <u>Inertial Range Summary</u>

   a. Alfven waves dominate cascade at $k_\perp \rho_i \ll 1$

   b. Turbulence undergoes a transition at $k_\perp \rho_i \sim 1$

   c. Scale-dependent Anisotropy determined by critical balance.

Dissipation Range

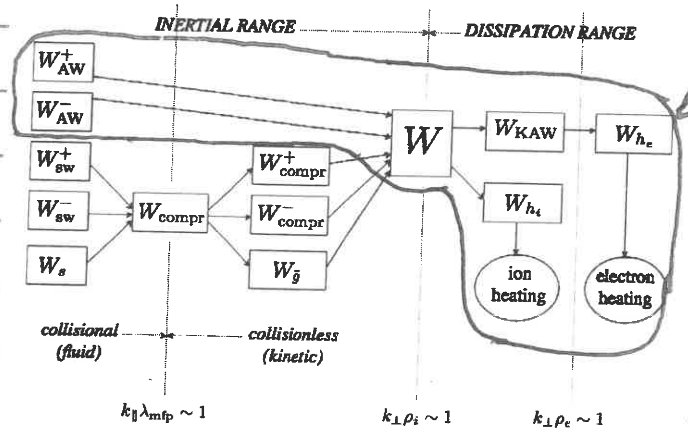
1. At $k_\perp \rho_i \sim 1$, the Alfvenic and compressible fluctuations couple together, and a kinetic description is essential to fully understand the dynamics

2. The Alfven wave fluctuations of the inertial range couple to two separate cascades at $k_\perp \rho_i > 1$:

   a. Kinetic Alfven Wave (KAW) Cascade

   b. Ion Entropy Cascade.

3. Summary of Kinetic Turbulent Cascade of Energy
Ref: Fig 5 of the Tome.



We'll focus on the Alfvénic inertial range cascade and the full dissipation range cascade

4. In the remainder of this lecture, we'll focus on developing the scaling theory and a cascade model of the Alfven and Kinetic Alfven wave cascades:

a. Assume Kolmogorov's Hypothesis of Local interactions

b. Extend Critical Balance to Kinetic Scales.

II. Scaling Theory and Cascade Model for Kinetic Turbulence

A. Scaling Theory:

1. Assume $\boxed{\text{Kolmogorov Hypothesis}}$:
   a. Local interactions (in scale)

   b. Constant Energy Cascade Rate: $\epsilon_0 = \frac{\text{Energy}}{\text{Time}} \propto \frac{v^2}{\tau}$

2. Since it is the magnetic energy spectrum that is observable, we follow the flow of magnetic energy

   a. Define $\boxed{b_k \equiv \frac{\delta B_\perp(k_\perp)}{\sqrt{4\pi m_i n_i}}}$ (cgs units)

   $\uparrow$
   Perpendicular Magnetic Field perturbation in velocity units

   b. Perpendicular magnetic energy density is $\frac{\delta B_\perp^2(k_\perp)}{8\pi}$

   c. Two-fluid velocities: At scales $k_\perp \rho_i > 1$, the ions decouple from the field motions, so it is better to use magnetic field fluctuations than velocity fluctuations.

3. Scaling theory: a. $E = \frac{1}{2}\rho v^2$ is kinetic energy density.
   b. We'll assume ρ = constant, and drop $\frac{\rho}{2}$, so $E \propto v^2$

   c. Also, define $\boxed{v_k = v_\perp(k_\perp)}$

Lecture #11 II. A. (Continued)

4. Local nonlinear cascade rate: $\boxed{\omega_{nl} \sim k_\perp v_k}$

a. This is an angular frequency, but in scaling theory we drop factors of 2π, $\omega_{nl} \sim \frac{1}{\tau_{nl}}$

5. Critical Balance:

a. We assume a critical balance of linear and nonlinear timescales (frequencies) at all scales!

$$\boxed{\omega \sim \omega_{nl}}$$

6. Linear Alfven and Kinetic Alfven Wave Frequency:

a. $\boxed{\omega = k_\parallel v_A \bar{\omega}}$ where $\boxed{\bar{\omega} = \sqrt{1 + \frac{(k_\perp \rho_i)^2}{\beta_i + \frac{2}{(1+T_e/T_i)}}}}$

b. Since β_i and $\frac{T_i}{T_e}$ are constant for a given plasma, the scaling is

$$\bar{\omega} \propto \begin{cases} 1 & k_\perp \rho_i \ll 1 \quad \text{MHD Regime} \\ k_\perp & k_\perp \rho_i \gg 1 \quad \text{KAW Regime (Dispersive)} \end{cases}$$

7. Eigenfunction relation between v_k and b_k (for Alfven waves)

a. $\boxed{v_k = \pm \bar{\omega} b_k}$

↑ upward or downward propagating wave.

b. For $k_\perp \rho_i \gg 1$, $v_k \propto k_\perp b_k$

↖ ↗ Different Scaling.

Lecture #11 II. A. (Continued)

8. Cascade Rate and Scaling:

a. $\epsilon_0 \sim \frac{\text{Energy}}{\text{Time}} \sim \frac{E}{\tau} \sim \frac{b_k^2}{\tau_{nl}} \sim \omega_{nl} b_k^2$

b. $\epsilon_0 \sim \omega_{nl} b_k^2 \sim (k_\perp v_k) b_k^2 \sim k_\perp \bar{\omega} b_k^3$

c. Scaling for b_k: $b_k \sim \epsilon_0^{\frac{1}{3}} k_\perp^{-\frac{1}{3}} \bar{\omega}^{-\frac{1}{3}}$

d. Limits:

$$b_k \propto \begin{cases} k_\perp^{-\frac{1}{3}} & k_\perp \rho_i \ll 1 \quad \text{MHD} \\ k_\perp^{-\frac{2}{3}} & k_\perp \rho_i \gg 1 \quad \text{KAW} \end{cases}$$

9. Energy Spectrum: $E_B(k_\perp) = \frac{b_k^2}{k_\perp}$

a. $E_B(k_\perp) \sim \frac{\epsilon_0^{2/3} k_\perp^{-2/3} \bar{\omega}^{-2/3}}{k_\perp} \sim \epsilon_0^{2/3} k_\perp^{-5/3} \bar{\omega}^{-2/3}$

b. Limits

$$E_B(k_\perp) \propto \begin{cases} k_\perp^{-5/3} & k_\perp \rho_i \ll 1 \quad \text{MHD} \\ k_\perp^{-7/3} & k_\perp \rho_i \gg 1 \quad \text{KAW} \end{cases}$$

10. Anisotropy (Critical Balance)

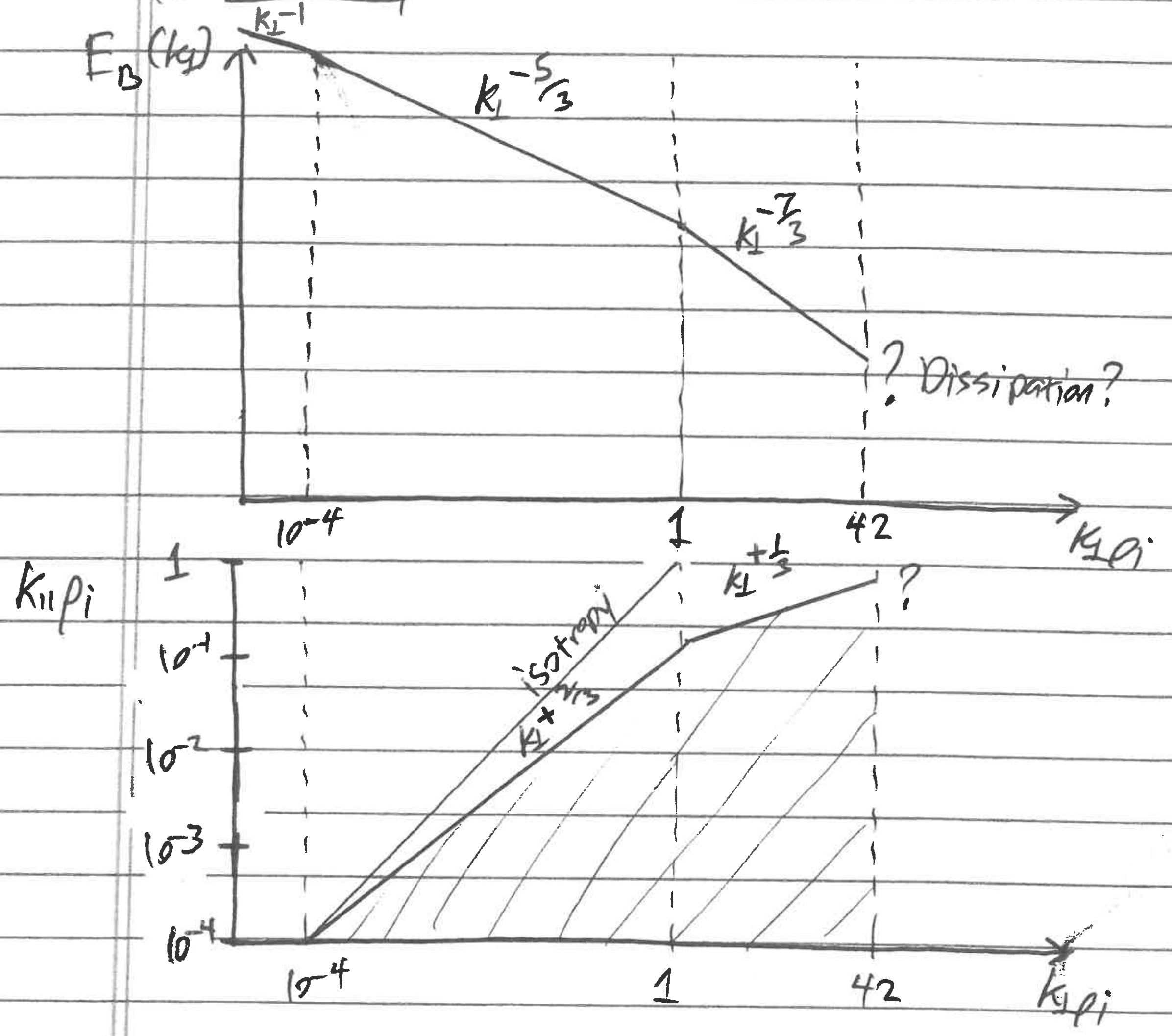
a. $\omega \sim \omega_{nl} \Rightarrow k_\parallel v_A \bar{\omega} \sim k_\perp \bar{\omega} b_k$

b. $k_\parallel \sim k_\perp \frac{(\epsilon_0^{\frac{1}{3}} k_\perp^{-\frac{1}{3}} \bar{\omega}^{-\frac{1}{3}})}{v_A} \propto k_\perp^{\frac{2}{3}} \bar{\omega}^{-\frac{1}{3}}$

c. Limits:

$$k_\parallel \propto \begin{cases} k_\perp^{2/3} & k_\perp \rho_i \ll 1 \quad \text{MHD} \\ k_\perp^{\frac{1}{3}} & k_\perp \rho_i \gg 1 \quad \text{KAW} \end{cases}$$

1. Summary:


$k_\perp^{-1}$
$E_B(k_\perp)$
$k_\perp^{-5/3}$
$k_\perp^{-7/3}$
? Dissipation?
10^{-4}
1
42
$k_\perp \rho_i$
$k_\parallel \rho_i$
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
isotropy
$k_\perp^{+2/3}$
$k_\perp^{+1/3}$
?
10^{-4}
1
42
$k_\perp \rho_i$

Lecture #11: II. (Continued)

B. Cascade Model

Refs: Howes et al. JGR (2008)
Howes, TenBarge, Dorland, PoP (2011)

1. The scalings predicted in Sec II A. assumption no dissipation in the cascade.

2. In reality, we expect collisionless damping may remove energy at a rate depending on scale $k_\perp$, competing with the nonlinear cascade to smaller scales.

3. A simple 1D (in $k_\perp$) Cascade Model can help to numerically assess the effect of this damping on the turbulent energy spectrum and anisotropy.

4. Accounting for Damping:

a. Assume the energy cascade rate may ~~now~~ decrease with scale, $\epsilon = \epsilon(k_\perp)$.

$$\Rightarrow \boxed{\epsilon(k_\perp) \sim k_\perp \bar{\omega}\, b_k^3}$$

b. Collisionless damping (e.g. Landau damping) can remove energy at a rate $\gamma(k_\perp)$

5. Energy Cascade Model:

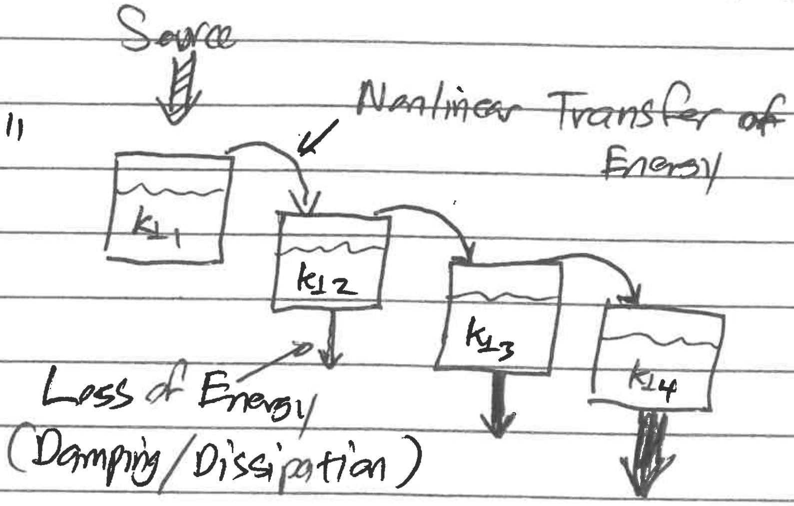
a. $$\frac{\partial b_k^2}{\partial t} = \underbrace{-k_\perp \frac{\partial \epsilon(k_\perp)}{\partial k_\perp}}_{\text{Nonlinear cascade scale to scale}} + \underbrace{S(k_\perp)}_{\text{Energy Source at large scales}} - \underbrace{2\gamma b_k^2}_{\text{Rate of energy loss by damping}}$$

<u>Lecture #11:</u> II. B5. (Continued)

b. Equation can be solved numerically to obtain a steady state, $\frac{\partial b_k^2}{\partial t} = 0$

c. Resulting solution for $b_k(k_\perp)$ gives $E_B(k_\perp)$ & $k_\parallel(k_\perp)$

6. "Bucket Analogy"



7. Linear Dispersion relation can be solved to obtain an estimate for $\gamma(k_\perp)$ in the turbulent plasma.

8. Further refinements of the cascade model include

<u>Ref:</u> Weakened Cascade Model, Howes, TenBarge & Dorland PoP (2011)

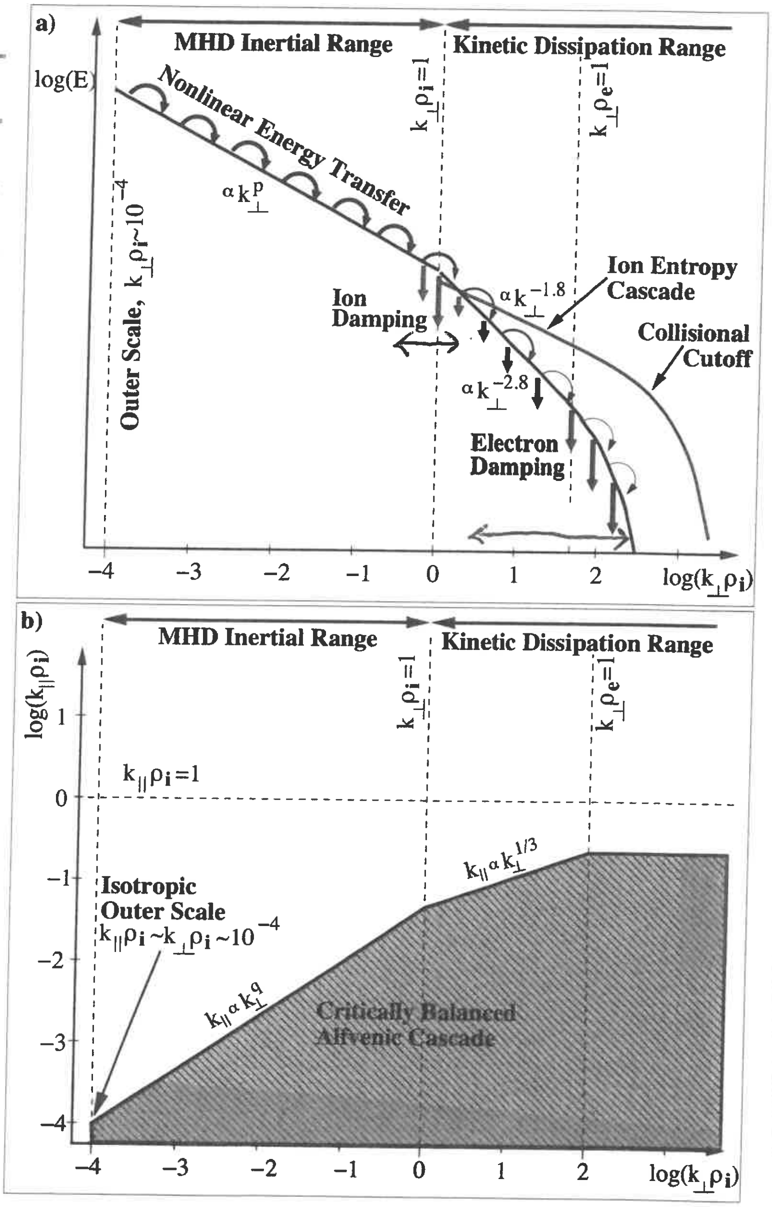
a. Incorporate both <u>weak and strong turbulence</u>

b. <u>Nonlocal interactions</u> contributing to cascade

⇒ Nonlocal interactions, in particular, prevent sharp cut offs of spectrum, bringing results into better agreement with simulations and observations.

Lecture # II. (Continued)

C. Summary of Kinetic Turbulence Scaling Theories & Cascade Models



Ref: "Kinetic Turbulence", Chapter in "Magnetic Fields in Diffuse Media"
Howes, G. , editors: Lazarian, de Gouveia Dal Pino, Melioli
ArXiV: 1502.04130