Lecture #13: Key Results on Kinetic Turbulence: Inertial Range

Ref: Howes, G.G. "Kinetic Turbulence" in Magnetic Fields in Diffuse Media, ed. de Gouveia Dal Pino & Lazarian, Springer: Heidelberg (2015)

I. The Inertial Range of the Kinetic Turbulence Cascade

A. The Initiation of the Turbulent Cascade

1. Whatever the supply of energy at large scales (either decaying large-scale fluctuations in the energy containing range, or driven fluctuations at a large scale), when fluctuations at a given scale $L \sim \frac{1}{k_0}$ have evolved for a nonlinear cascade time $\tau_{nl} \sim \frac{1}{\omega_{nl}}$, the fluctuations can nonlinearly transfer their energy to smaller scales, initiating the turbulent cascade.

2. "Outer Scale" of turbulence:
   a. Typically large scale $\boxed{k_0 \rho_i \ll 1}$ $\Rightarrow \frac{\rho_i}{L} \ll 1$

   b. Thus MHD is a useful starting point for inertial range physics. Typically $k_0\rho_i \sim 10^{-3}$ to 10^{-4} in the near-Earth solar wind

3. Strong Turbulence
   a. Lecture #8 - Weak MHD Turbulence
   b. Lecture #9 - Strong MHD Turbulence

$\Rightarrow$ Turbulence generally transitions to strong turbulence, so we'll focus here on strong turbulence ($\chi \sim 1$) at outer scale.

I. A. (Continued)

4. At the outer scale, the turbulent fluctuations are a mixture of

Three propagating MHD wave modes: Alfven waves → Incompressible
Fast magnetosonic waves } Compressible
Slow magnetosonic waves }

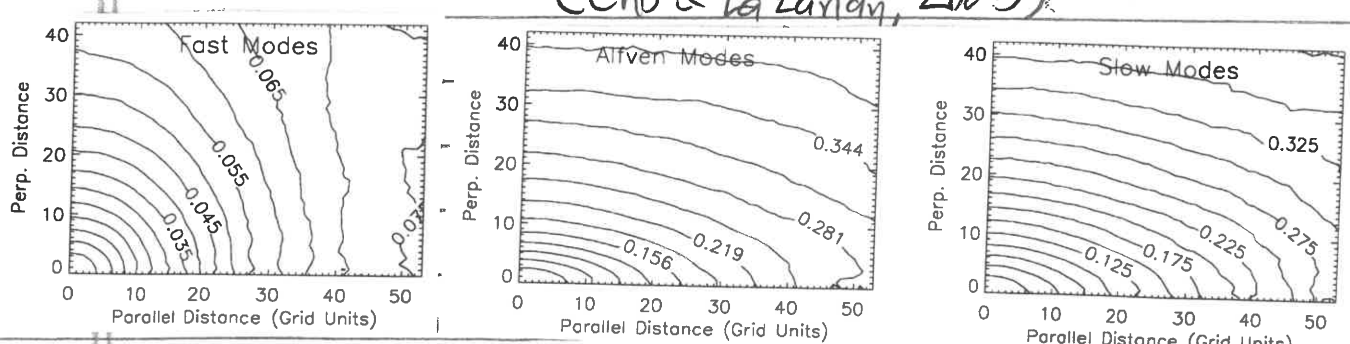
Non-propagating ($\omega = 0$): Entropy Mode → Compressible

B. Turbulent Cascade

1. Early numerical results and analytical theory established some important properties of MHD turbulence.

2.a. $\underbrace{\text{Fast waves}}_{\text{isotropic cascade}}$ decouple from $\underbrace{\text{Alfven, slow, \& entropy modes}}_{\text{anisotropic cascade}}$

(Cho & Lazarian, 2003)



b. Entropy modes follow the Alfvenic cascade like a passive scalar (Maron & Goldreich, 2001)

3. Isotropic Fast Wave Cascade

a. Magnetic Energy Spectrum: $E_B(k) \propto k^{-\frac{3}{2}}$ (Cho & Lazarian, 2003)

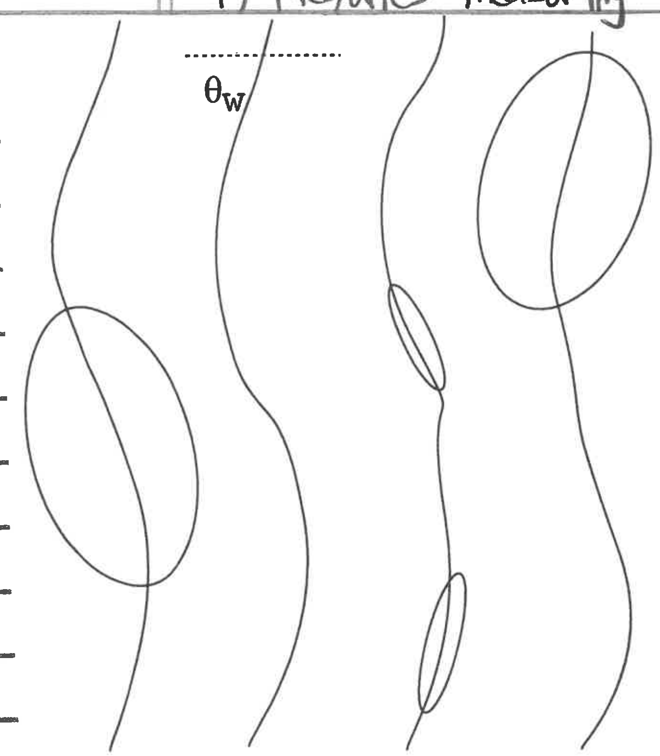
4. Anisotropic Alfvenic Cascade

a. Energy Spectrum: $E_B(k_\perp) \propto k_\perp^{\,p}$ $\begin{cases} p = -5/3 & \text{GS95 (Goldreich \& Sridhar, 1995)} \\ p = -\frac{3}{2} & \text{B06 (Boldyrev, 2006)} \end{cases}$

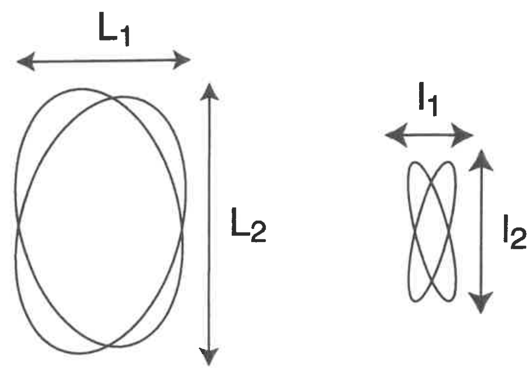
b. Wavevector Anisotropy: $k_\parallel = k_0^{1-q} k_\perp^{\,q}$ $\begin{cases} q = \frac{2}{3} & \text{GS95} \\ q = \frac{1}{2} & \text{B06} \end{cases}$

c. Numerical Measurement of Anisotropy:

i) Requires measuring δB relative to the local (not global) magnetic field.



(a)



(b)

(Cho & Vishniac, 2000)

ii) Fourier Analysis will fail (global)

iii) Requires projection relative local B $\Rightarrow$ structure functions can work.

4. (Continued)

d. Slow waves & entropy mode fluctuations are passively cascaded by the Alfvén waves.
(Maron & Goldreich, 2001) (Lithwick & Goldreich, 2001)

e. Alfven waves (and passive slow waves) obey critical balance.

$k_{\parallel} \propto k_{\perp}^{2/3}$ (Maron & Goldreich, 2001; Cho, Lazarian, & Vishniac, 2002; Cho & Vishniac, 2003)

5. Why do fast waves decouple from Alfven & slow waves?

a. Anisotropy of Alfvenic cascade leads to $k_{\parallel} \ll k_{\perp}$ at small scales.

b. Fast Wave: $\omega \simeq k\sqrt{v_A^2 + c_s^2} \simeq k v_A \sqrt{1+\beta_i\left(1+\frac{T_e}{T_i}\right)}$

Alfven Wave: $\omega = k_{\parallel} v_A$

(Howes, Klein, & TenBarge, 2014)

c. Frequency mismatch in anisotropic limit $k_{\parallel} \ll k_{\perp}$

$$\frac{\omega_F}{\omega_A} = \frac{k v_A \sqrt{1+\beta_i\left(1+\frac{T_e}{T_i}\right)}}{k_{\parallel} v_A} = \frac{k}{k_{\parallel}}\left[1+\beta_i\left(1+\frac{T_e}{T_i}\right)\right]^{\frac{1}{2}} \geq \frac{k_{\perp}}{k_{\parallel}} \gg 1$$

$\Rightarrow \omega_F \gg \omega_A$

With vastly different frequencies, Alfven waves and fast waves will not efficiently exchange energy.
(Lithwick & Goldreich, 2001; Howes et al, 2012)

C. Nonlinear Physics of Alfven Wave Cascade

1. Incompressible MHD is the simplest model that reproduces anisotropic, magnetized plasma turbulence
   a. Fast waves are excluded
   b. Alfven waves and pseudo-Alfven waves (the incompressible limit of slow waves) are included.

2. Equations of Incompressible MHD (Elsässer Form)

a. $$\frac{\partial \underset{\sim}{z}^{\pm}}{\partial t} \mp \underset{\sim}{V_A} \cdot \nabla \underset{\sim}{z}^{\pm} = -\underset{\sim}{z}^{\mp} \cdot \nabla \underset{\sim}{z}^{\pm} - \frac{\nabla P}{\rho_0}$$

b. $$\nabla \cdot \underset{\sim}{z}^{\pm} = 0$$

c. where ~~Elsasser~~ field $\underset{\sim}{z}^{\pm} = \underset{\sim}{u} \pm \frac{\delta \underset{\sim}{B}}{\sqrt{4\pi\rho_0}}$, $\underset{\sim}{V_A} = \frac{\underset{\sim}{B_0}}{\sqrt{4\pi\rho_0}}$

$\underset{\sim}{z}^{\pm}$ Represents Alfven waves propagating up/down magnetic field.

3. Linear and Nonlinear Physics of each term

$$\frac{\partial \underset{\sim}{z}^{\pm}}{\partial t} \underbrace{\mp \underset{\sim}{V_A} \cdot \nabla \underset{\sim}{z}^{\pm}}_{\text{Linear wave propagation up/down } \underset{\sim}{B_0}} = \underbrace{-\underset{\sim}{z}^{\mp} \cdot \nabla \underset{\sim}{z}^{\pm}}_{\text{Nonlinear interaction between upward \& downward Alfven waves}} - \underbrace{\frac{\nabla P}{\rho_0}}_{\text{Nonlinear term maintains incompressibility}}$$

⇒ Requires non-zero $k_\parallel$

⇒ Requires both components perpendicular to $\underset{\sim}{B_0}$

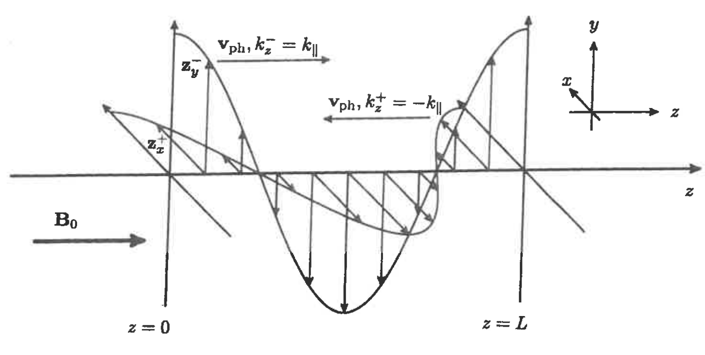
4. Early MHD turbulence studies recognized that interactions between Counterpropagating Alfven waves governed the turbulence cascade (Kraichnan, 1965)
$\Rightarrow$ known as "<u>Alfven Wave collisions</u>"

5. a. Weak MHD turbulence theory was based on these Alfven wave collisions (See Lec #2) (Sridhar & Goldreich, 1994)

b. Controversy about 3- vs. 4-wave interactions arose (See Sec II. B. 3 of Lec #2)
(Shebalin, Matthaeus, & Montgomery, 1983; Montgomery & Matthaeus, 1995; Ng & Bhattacharjee, 1996; Goldreich & Sridhar, 1997; Galtier, et al, 2000; Lithwick & Goldreich, 2003)

6. Howes & Nielson (2013) solved for the energy transfer in Alfven wave collisions analytically (Howes & Nielson, 2013) and numerically (Nielson, Howes, & Dorland, 2013)

a. <u>Set-up</u>: Two Counterpropagating, Perpendicularly polarized Alfven Waves.



$\Rightarrow$ Periodic BCs.

$$\underset{\sim}{k_1^+} = k_\perp \hat{\underset{\sim}{x}} - k_\parallel \hat{\underset{\sim}{z}}$$

$$\underset{\sim}{k_1^-} = k_\perp \hat{\underset{\sim}{y}} + k_\parallel \hat{\underset{\sim}{z}}$$

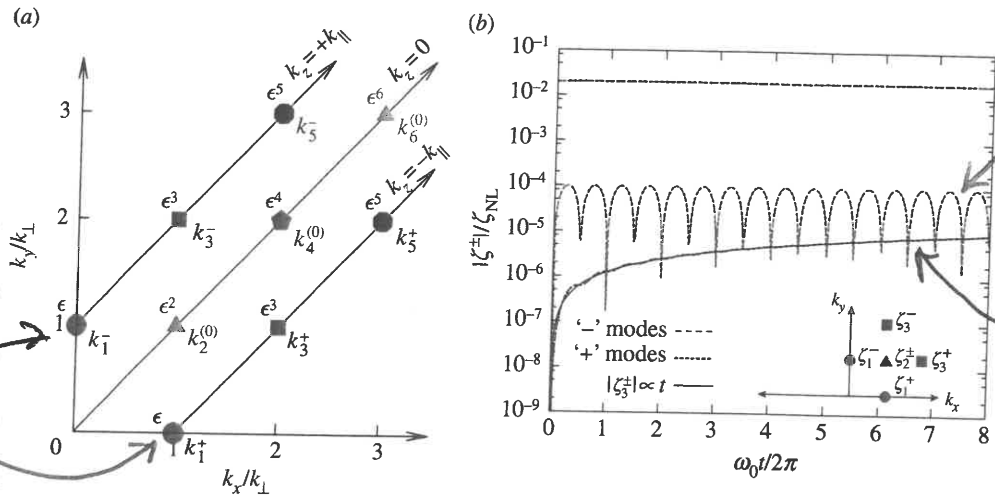
I. C. (Continued)

6. (Continued)

b. Energy Transfer:

In the weakly nonlinear limit ($\chi = \frac{k_\perp \underset{\sim}{z}^{\pm}}{k_\parallel v_A} \ll 1$),

Initial Alfven waves



$\underset{\sim}{k}_2^{(0)}$ does not gain net energy

$\underset{\sim}{k}_3^{\pm}$ modes secularly gain energy

i) First $\underset{\sim}{k}_1^+$ & $\underset{\sim}{k}_1^-$ interact nonlinearly to generate a secondary mode $\underset{\sim}{k}_2^{(0)} = k_\perp \hat{\underset{\sim}{x}} + k_\perp \hat{\underset{\sim}{y}}$

$\Rightarrow$ This $\underset{\sim}{k}_2^{(0)}$ mode is essentially a shear in the magnetic field.

ii) Next, $\underset{\sim}{k}_1^+$ & $\underset{\sim}{k}_1^-$ each interact with $\underset{\sim}{k}_2^{(0)}$ to transfer energy secularly to

$$\underset{\sim}{k}_3^+ = 2k_\perp \hat{\underset{\sim}{x}} + k_\perp \hat{\underset{\sim}{y}} - k_\parallel \hat{\underset{\sim}{z}}$$

$$\underset{\sim}{k}_3^- = k_\perp \hat{\underset{\sim}{x}} + 2k_\perp \hat{\underset{\sim}{y}} + k_\parallel \hat{\underset{\sim}{z}}$$

$\Rightarrow$ $\boxed{k_\parallel \text{ remains constant, } k_\perp \text{ increases}}$

I. C. (Continued)
6. (Continued)

c. Repeating: $k_{\parallel}$ remains constant ⇐ As expected for weak MHD turbulence

$k_{\perp}$ increases ⇐ $\boxed{\text{This is the energy cascade to smaller perpendicular scales}}$

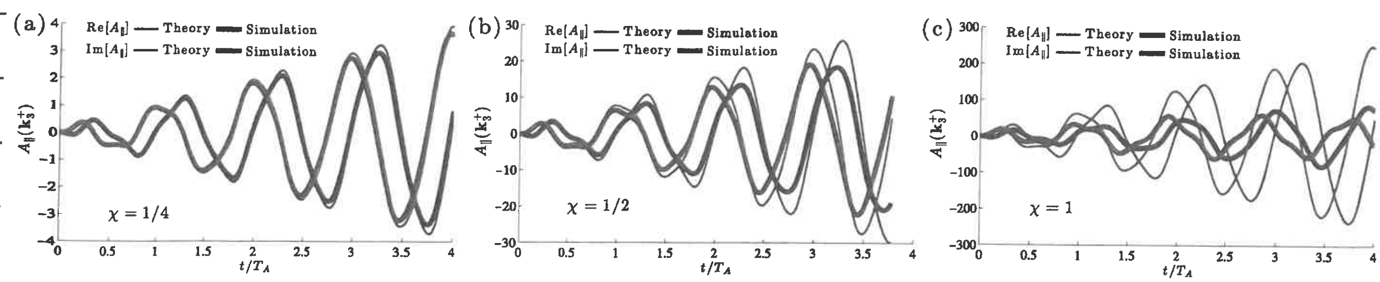
d. The physics of this energy transfer has been verified experimentally in UCLA's Large Plasma Device (LAPD) (Drake et al. 2013; Howes et al. 2012)

e. $\boxed{\text{Alfven Wave Collisions are the fundamental building block of astrophysical plasma turbulence.}}$

(Howes & Nielson, 2013; Howes et al. 2012; Howes, 2015)

7. Additional Results for Alfven Wave Collisions

a. The physics of nonlinear energy transfer persists qualitatively in the limit of strong turbulence, $\chi \to 1$.



(Howes, 2016)

b. In the limit of strong turbulence, phase and amplitude relations among primary & nonlinearly generated modes yield current sheets (another topic) (Howes, 2016; Verniero, Howes, & Klein 2018)

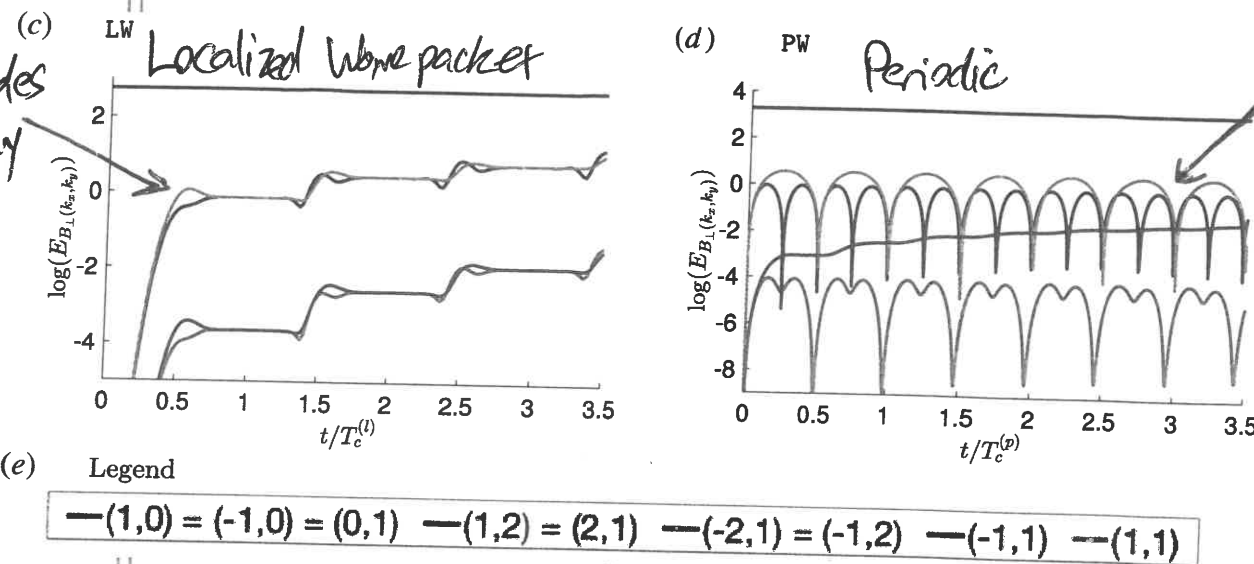
I. C. (Continued)

7. (Continued)

c. Under more realistic, non-periodic conditions, the secondary modes $\underset{\sim}{k}_2^{(0)}$ are simply Alfven waves (Verniero & Howes, 2018)

d. These secondary modes also secularly gain energy in the localized wavepacket (non-periodic) case.

$\underset{\sim}{k}_2^{(0)}$ modes secularly gain energy



$\underset{\sim}{k}_2^{(0)}$ modes do not gain net energy.

(Verniero & Howes, 2018)

8. What remains to be done to connect Alfven Wave Collisions to Alfvenic turbulence?

a. Connect physics of many nonlinearly interacting wavepackets to the resulting energy spectrum

b. Establish connection between Alfven wave collisions and the physics of dynamic alignment (in B06).

D. Collisional to Collisionless Transition

1. MHD is formally valid in the strongly collisional limit, $k_\parallel \lambda_i \sim \frac{\lambda_i}{\ell} \ll 1$
$\ell \leftarrow$ scale of fluct variations.

a. Note that λ_i is the ion collisional mean free path.

b. Due to the magnetic field, ions can only travel a distance $\ell_\perp \sim \rho_i$ ~~from~~ perpendicular to $\underset{\sim}{B}_0$, so the collisional transition relates to parallel direction, $\ell_\parallel \sim \frac{1}{k_\parallel}$.

c. The magnetic field effectively makes perpendicular motions fluid-like.

2. At $k_\parallel \lambda_i \sim 1$, turbulence transitions from collisional (MHD) to collisionless (requiring kinetics).

3. Because fast waves are isotropic & Alfven waves anisotropic, the perpendicular wavenumber $k_\perp$ associated with $k_\parallel \lambda_i \sim 1$ differs:

a. Alfven waves: $k_\parallel = k_0^{1-q} k_\perp^q = k_0 \left(\frac{k_\perp}{k_0}\right)^q$

$$\Rightarrow k_\perp = k_0 \left(\frac{k_\parallel}{k_0}\right)^{\frac{1}{q}} = k_0 \frac{(\cancel{k_\parallel \lambda_i})^{\frac{1}{q}}}{(k_0 \lambda_i)^{\frac{1}{q}}} = \boxed{k_0 (k_0 \lambda_i)^{-\frac{1}{q}} = k_{\perp c}}$$

b. Fast waves: $k_\parallel = k_\perp \Rightarrow \boxed{k_{\perp c} \lambda_i = 1}$

c. Since $q < 1$, fast waves reach collisional transition first.

4. Where does collisionless transition $k_{\perp c}$ fall?

a. In cold astrophysical plasmas at very large scales [e.g., "The Great Power law in the Sky", (Armstrong, Rickett, & Spangler, 1995)], the collisional transitions falls in the inertial range, $k_{\perp c} > k_0$ and $k_{\perp c}\rho_i \ll 1$.

$$\left(\text{or } \frac{1}{k_0} > \frac{1}{k_{\perp c}} > \frac{1}{\rho_i}\right)$$

b. In the hot and diffuse solar wind, $k_{\perp c} < k_0$, so the entire inertial range is weakly collisional to collisionless.

$$\lambda_i \sim 1\,AU \sim 1.5\times 10^{13}\,cm$$
$$\frac{1}{k_0} \sim 10^{11} \text{ to } 10^{12}\,cm.$$

5. What happens to the cascade at $k_{\perp c}$?

a. For <u>compressible waves</u> (fast waves, slow waves, entropy modes), fluctuations suffer strong collisional damping by ion viscosity at $k_\parallel \lambda_i \sim 1$ (or $k_\perp \sim k_{\perp c}$) (Braginskii, 1965)

b. Any compressible energy passing through this transition is expected to manifest as the kinetic counterparts of the MHD fast & slow waves (e.g., Klein et. al. (2012)) for weakly collisional conditions at $k_\perp \gg k_{\perp c}$ (Schekochihin et al. 2009)

I. D. (Continued)

5. (Continued)

c. Alfven Waves:

i. Alfven waves are incompressible, with no motions along B_0

ii. Thus, they are unaffected by the collisionless transition and continue undamped down to $k_\perp \rho_i \sim 1$. (Schekochihin, et al., 2009)

E. Inertial Range: Between Collisional Transition & Ion Scales

1. Anisotropic Fluctuations: $k_\parallel \ll k_\perp$

a. Critical balance predicts $\frac{k_\parallel}{k_\perp} = \left(\frac{k_0}{k_\perp}\right)^{1-q}$

b. Since $q < 1$ for either GS95 or B06,

when $k_\perp \gg k_0$ (small scales compared to outer scale)

we have $\boxed{k_\parallel \ll k_\perp}$

c. In this $k_\parallel \ll k_\perp$ limit:

(i) Kinetic dynamics of anisotropic Alfvénic fluctuations is rigorously described by reduced MHD (Strauss, 1976)

(ii) Alfven wave cascade is undamped down to $k_\perp \rho_i \sim 1$

(iii) Slow & Alfven wave cascades don't exchange energy

(iv) fast waves don't interact due to mismatch in frequency.

"Schekochihin, et al. 2009) ⇒ "The Tome"

E. Summary of Inertial Range in Kinetic Turbulence

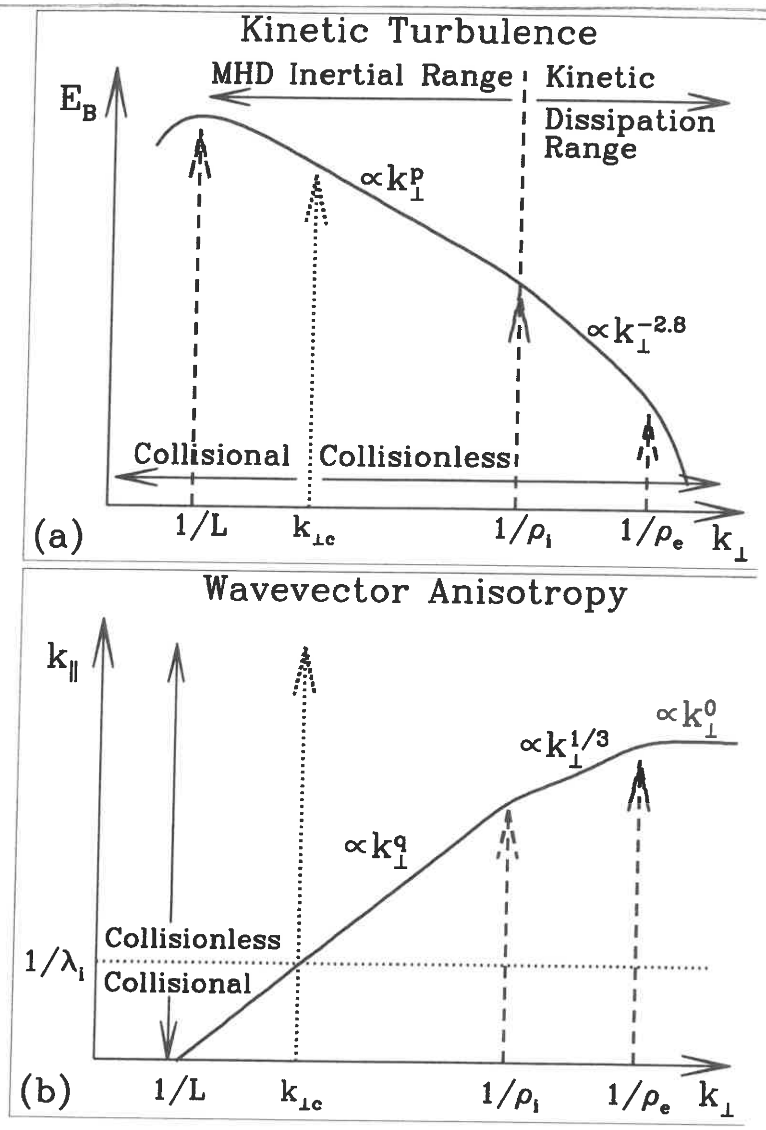


Fig. 2 (a) Perpendicular wavenumber spectrum for magnetic energy in kinetic turbulence, from the driving scale, L, through the MHD inertial range to the ion Larmor radius ρ_i, where the turbulent cascade enters the kinetic dissipation range, and down to the electron Larmor radius ρ_e. The transition from collisional to collisionless dynamics occurs at $k_{\perp c}$. (b) Wavevector anisotropy in kinetic turbulence, scaling as $k_\perp^q$ in the MHD inertial range, $k_\perp^{1/3}$ in the kinetic dissipation range, and $k_\perp^0$ (no parallel cascade) beyond electron scales. The transition from collisional to collisionless dynamics occurs at $k_\parallel \rho_i \sim 1$.

Bibliography

Boldyrev, S. (2006). Spectrum of Magnetohydrodynamic Turbulence. *Phys. Rev. Lett.*, 96(11):115002.

Braginskii, S. I. (1965). Transport Processes in a Plasma. *Rev. Plasma Phys.*, 1:205–+.

Cho, J. and Lazarian, A. (2003). Compressible magnetohydrodynamic turbulence: mode coupling, scaling relations, anisotropy, viscosity-damped regime and astrophysical implications. *Mon. Not. Roy. Astron. Soc.*, 345:325–339.

Cho, J., Lazarian, A., and Vishniac, E. T. (2002). Simulations of Magnetohydrodynamic Turbulence in a Strongly Magnetized Medium. *Astrophys. J.*, 564:291–301.

Cho, J. and Vishniac, E. T. (2000). The Anisotropy of Magnetohydrodynamic Alfvénic Turbulence. *Astrophys. J.*, 539:273–282.

Drake, D. J., Schroeder, J. W. R., Howes, G. G., Kletzing, C. A., Skiff, F., Carter, T. A., and Auerbach, D. W. (2013). Alfvén wave collisions, the fundamental building block of plasma turbulence. IV. Laboratory experiment. *Phys. Plasmas*, 20(7):072901.

Galtier, S., Nazarenko, S. V., Newell, A. C., and Pouquet, A. (2000). A weak turbulence theory for incompressible magnetohydrodynamics. *J. Plasma Phys.*, 63:447–488.

Goldreich, P. and Sridhar, S. (1995). Toward a Theory of Interstellar Turbulence II. Strong Alfvénic Turbulence. *Astrophys. J.*, 438:763–775.

Howes, G. G. (2014). The inherently three-dimensional nature of magnetized plasma turbulence. *Journal of Plasma Physics*, FirstView:1–19.

Howes, G. G. (2015). A dynamical model of plasma turbulence in the solar wind. *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 373(2041):20140145.

Howes, G. G. (2015). *Magnetic Fields in Diffuse Media*, chapter Kinetic Turbulence. Springer, Heidelberg.

Howes, G. G., Bale, S. D., Klein, K. G., Chen, C. H. K., Salem, C. S., and TenBarge, J. M. (2012). The Slow-mode Nature of Compressible Wave Power in Solar Wind Turbulence. *Astrophys. J. Lett.*, 753:L19.

Howes, G. G., Klein, K. G., and TenBarge, J. M. (2014). Validity of the Taylor Hypothesis for Linear Kinetic Waves in the Weakly Collisional Solar Wind. *Astrophys. J.*, 789:106.

Howes, G. G. and Nielson, K. D. (2013). Alfvén wave collisions, the fundamental building block of plasma turbulence. I. Asymptotic solution. *Phys. Plasmas*, 20(7):072302.

Klein, K. G., Howes, G. G., TenBarge, J. M., Bale, S. D., Chen, C. H. K., and Salem, C. S. (2012). Using Synthetic Spacecraft Data to Interpret Compressible Fluctuations in Solar Wind Turbulence. *Astrophys. J.*, 755:159.

Kraichnan, R. H. (1965). Inertial range spectrum of hyromagnetic turbulence. *Phys. Fluids*, 8:1385–1387.

Lithwick, Y. and Goldreich, P. (2001). Compressible magnetohydrodynamic turbulence in interstellar plasmas. *Astrophys. J.*, 562:279–296.

Lithwick, Y. and Goldreich, P. (2003). Imbalanced weak magnetohydrodynamic turbulence. *Astrophys. J.*, 582:1220–1240.

Maron, J. and Goldreich, P. (2001). Simulations of incompressible magnetohydrodynamic turbulence. *Astrophys. J.*, 554:1175–1196.

Montgomery, D. and Matthaeus, W. H. (1995). Anisotropic Modal Energy Transfer in Interstellar Turbulence. *Astrophys. J.*, 447:706.

Ng, C. S. and Bhattacharjee, A. (1996). Interaction of Shear-Alfven Wave Packets: Implication for Weak Magnetohydrodynamic Turbulence in Astrophysical Plasmas. *Astrophys. J.*, 465:845.

Nielson, K. D., Howes, G. G., and Dorland, W. (2013). Alfvén wave collisions, the fundamental building block of plasma turbulence. II. Numerical solution. *Phys. Plasmas*, 20(7):072303.

Schekochihin, A. A., Cowley, S. C., Dorland, W., Hammett, G. W., Howes, G. G., Quataert, E., and Tatsuno, T. (2009). Astrophysical Gyrokinetics: Kinetic and Fluid Turbulent Cascades in Magnetized Weakly Collisional Plasmas. *Astrophys. J. Supp.*, 182:310–377.

Shebalin, J. V., Matthaeus, W. H., and Montgomery, D. (1983). Anisotropy in mhd turbulence due to a mean magnetic field. *J. Plasma Phys.*, 29:525–547.

Sridhar, S. and Goldreich, P. (1994). Toward a theory of interstellar turbulence. 1: Weak Alfvenic turbulence. *Astrophys. J.*, 432:612–621.

Strauss, H. R. (1976). Nonlinear, three-dimensional magnetohydrodynamics of noncircular tokamaks. *Phys. Fluids*, 19:134–140.

Verniero, J. L. and Howes, G. G. (2018). The Alfvénic nature of energy transfer mediation in localized, strongly nonlinear Alfvén wavepacket collisions. *J. Plasma Phys.*, 84(1):905840109.

Verniero, J. L., Howes, G. G., and Klein, K. G. (2018). Nonlinear energy transfer and current sheet development in localized Alfvén wavepacket collisions in the strong turbulence limit. *J. Plasma Phys.*, 84(1):905840103.