Lecture #16: MHD Shocks & Discontinuities I

I. Conditions on Fluid Boundaries in MHD

A. Fluid Boundaries

1. Boundaries separate adjacent plasma regions
   a. These boundary layers can evolve to become very thin.
   b. Sufficiently thin layers can be treated as discontinuities

2. Discontinuities
   a. Outside of discontinuities, you can describe the separate regions using ideal MHD

   b. Fields and parameters across discontinuities cannot change arbitrarily — must satisfy constraints of the MHD equations.

   c. These constraints — the jump conditions — admit different classes of discontinuities.

3. Typically, the transition from one state to another requires dissipation
   a. Ex: Hydrodynamics: i) Euler equations in regions
   ii) Navier-Stokes equations in boundary layer

I. (Continued)

B. General Jump Conditions

1. Assume boundary layer is infinitesimally thin
   a. For MHD, $\Delta x \ll L \leftarrow$ macroscopic scale lengths.

   b. But, $\Delta x \gtrsim \rho_i$ ion Larmor radius
   $\Delta x \gtrsim \lambda_{De}$ Debye length

2. Strategy:
   ① Write equations of MHD in conservation law form
   ② In the shock rest frame, $\frac{\partial}{\partial t} = 0$
   ③ Integrate over discontinuity to obtain jump conditions
   ④ Combine equations for jump conditions to obtain a "linear dispersion relation" that define different classes of discontinuities

3. Start with the equations of ideal MHD

Ⓐ a. $\frac{\partial \rho}{\partial t} + \underset{\sim}{u} \cdot \nabla \rho = -\rho \nabla \cdot \underset{\sim}{u}$

Ⓑ b. $\rho \frac{\partial \underset{\sim}{u}}{\partial t} + \rho \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \frac{(\underset{\sim}{B} \cdot \nabla)\underset{\sim}{B}}{\mu_0}$

Ⓒ c. $\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{u} \times \underset{\sim}{B})$

Ⓓ d. $\frac{\partial p}{\partial t} + \underset{\sim}{u} \cdot \nabla p = -\gamma p \nabla \cdot \underset{\sim}{u}$

Ⓔ e. $\nabla \cdot \underset{\sim}{B} = 0$

I. B. (Continued) Hanes (3)

4. Manipulate equations into "Conservation Form", $\frac{\partial q}{\partial t} + \nabla \cdot \underset{\sim}{Q} = 0$

a. Mass Density

$$\frac{\partial \rho}{\partial t} + (\underset{\sim}{U} \cdot \nabla)\rho + \rho \nabla \cdot \underset{\sim}{U} = 0 \Rightarrow \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{U}) = 0}$$

i) NRL(7) $(\underset{\sim}{A} \cdot \nabla) f + f \nabla \cdot \underset{\sim}{A} = \nabla \cdot (f \underset{\sim}{A})$

ρ = mass density , $\rho \underset{\sim}{U}$ = mass density flux

b. Momentum Density:

$$\frac{\partial (\rho \underset{\sim}{U})}{\partial t} = \underbrace{\rho \frac{\partial \underset{\sim}{U}}{\partial t}}_{(B)} + \underbrace{\underset{\sim}{U} \frac{\partial \rho}{\partial t}}_{\underset{\sim}{U}(A)} = \underbrace{-\rho \underset{\sim}{U} \cdot \nabla \underset{\sim}{U}}_{(i)} - \nabla\left(p + \frac{B^2}{2\mu_0}\right) + \underbrace{\frac{(\underset{\sim}{B} \cdot \nabla)\underset{\sim}{B}}{\mu_0}}_{(iv)} \underbrace{- \underset{\sim}{U}(\underset{\sim}{U} \cdot \nabla)\rho}_{(ii)} \underbrace{- \rho \underset{\sim}{U} \nabla \cdot \underset{\sim}{U}}_{(iii)}$$

i) Using NRL(17) $\nabla \cdot (\underset{\sim}{A}\underset{\sim}{B}) = (\nabla \cdot \underset{\sim}{A})\underset{\sim}{B} + (\underset{\sim}{A} \cdot \nabla)\underset{\sim}{B}$

with $\underset{\sim}{A} = \rho \underset{\sim}{U}$ and $\underset{\sim}{B} = \underset{\sim}{U}$

(i), (ii), & (iii) become $= -\nabla \cdot (\rho \underset{\sim}{U}\underset{\sim}{U})$

ii) Same formula gives $\nabla \cdot (\underset{\sim}{B}\underset{\sim}{B}) = \cancel{(\nabla \cdot \underset{\sim}{B})}^{\,0}\,\underset{\sim}{B} + (\underset{\sim}{B} \cdot \nabla)\underset{\sim}{B}$

iii) Thus

$$\boxed{\frac{\partial (\rho \underset{\sim}{U})}{\partial t} + \nabla \cdot (\rho \underset{\sim}{U}\underset{\sim}{U}) = -\nabla\left(p + \frac{B^2}{2\mu_0}\right) + \nabla \cdot \left(\frac{\underset{\sim}{B}\underset{\sim}{B}}{\mu_0}\right)}$$

c. Induction Equation:

$$\boxed{\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B})}$$

Although not strictly in "conservation law form", we can use this form to easily obtain a jump condition

d. Divergence-free $\underset{\sim}{B}$: $\boxed{\nabla \cdot \underset{\sim}{B} = 0}$

e. Energy Density:

i) MHD Energy Density

$$\mathcal{E} = \underbrace{\frac{1}{2}\rho U^2}_{\text{Kinetic Energy Density}} + \underbrace{\frac{p}{\gamma-1}}_{\text{Internal (Thermal) Energy Density}} + \underbrace{\frac{B^2}{2\mu_0}}_{\text{Magnetic Energy Density}}$$

ii) NOTE: $\frac{p}{\gamma-1} = \frac{3}{2}p = \frac{3}{2}nkT$ for $\gamma = \frac{5}{3}$ (monatomic gas)

iii) $$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho U^2 + \frac{p}{\gamma-1} + \frac{B^2}{2\mu_0}\right) = \underbrace{\rho \underset{\sim}{U} \cdot \frac{\partial \underset{\sim}{U}}{\partial t}}_{B} + \underbrace{\frac{U^2}{2}\frac{\partial \rho}{\partial t}}_{A} + \underbrace{\frac{1}{\gamma-1}\frac{\partial p}{\partial t}}_{D} + \underbrace{\frac{1}{\mu_0}\underset{\sim}{B} \cdot \frac{\partial \underset{\sim}{B}}{\partial t}}_{C}$$

iv) After several pages of tedious algebra,

$$\boxed{\frac{\partial}{\partial t}\left(\frac{1}{2}\rho U^2 + \frac{p}{\gamma-1} + \frac{B^2}{2\mu_0}\right) + \nabla \cdot \left(\frac{1}{2}\rho U^2 \underset{\sim}{U} + \underbrace{\frac{\gamma p}{\gamma-1}\underset{\sim}{U}}_{\text{enthalpy flux}} + \frac{B^2}{\mu_0}\underset{\sim}{U} - \frac{1}{\mu_0}(\underset{\sim}{U} \cdot \underset{\sim}{B})\underset{\sim}{B}\right) = 0}$$

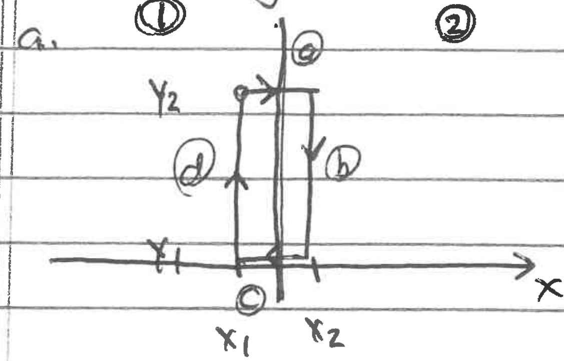
$\Rightarrow$ Step 1 completed.

5. In the shock rest frame, the shock structure is steady in time $\Rightarrow \frac{\partial}{\partial t} = 0$.

$$\boxed{\begin{array}{ll}
a. & \nabla \cdot (\rho \underset{\sim}{U}) = 0 \\
b. & \nabla \cdot (\rho \underset{\sim}{U}\underset{\sim}{U}) + \nabla\left(p + \frac{B^2}{2\mu_0}\right) - \nabla \cdot \left(\frac{\underset{\sim}{B}\underset{\sim}{B}}{\mu_0}\right) = 0 \\
c. & \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B}) = 0 \\
d. & \nabla \cdot \underset{\sim}{B} = 0 \\
e. & \nabla \cdot \left(\frac{1}{2}\rho U^2 \underset{\sim}{U} + \frac{\gamma p}{\gamma-1}\underset{\sim}{U} + \frac{B^2}{\mu_0}\underset{\sim}{U} - \frac{1}{\mu_0}(\underset{\sim}{U} \cdot \underset{\sim}{B})\underset{\sim}{B}\right) = 0
\end{array}}$$

6. Converting to Jump Equations

a.



Perform a line integral around a loop from ① to ③ and back.

b. Consider a general gradient: $\nabla A = \frac{\partial A}{\partial x}\hat{\underline{x}}$

shock

1D variation along normal

$$\frac{1}{2}\oint \nabla A\cdot d\underline{l} = \frac{\hat{\underline{x}}}{2}\cdot\oint \frac{\partial A}{\partial x}\, d\underline{l} = \frac{\hat{\underline{x}}}{2}\cdot\Big\{\int_{x_1}^{x_2}\frac{\partial A}{\partial x}(dx\,\hat{\underline{x}}) + \int_{y_2}^{y_1}\frac{\partial A}{\partial x}\cancel{(dy\,\hat{\underline{y}})} + \int_{x_2}^{x_1}\frac{\partial A}{\partial x}(-dx\,\hat{\underline{x}})$$

$$+ \int_{y_1}^{y_2}\frac{\partial A}{\partial x}\cancel{(dy\,\hat{\underline{y}})}\Big\} = \frac{1}{2}\int_{x_1}^{x_2}\frac{\partial A}{\partial x}dx - \frac{1}{2}\int_{x_2}^{x_1}\frac{\partial A}{\partial x}dx = \frac{1}{2}\int_{x_1}^{x_2}dA - \frac{1}{2}\int_{x_2}^{x_1}dA$$

$$= \frac{2}{2}\int_{x_1}^{x_2}dA = A(x_2) - A(x_1) \equiv [A]$$

b. Define: Jump $[A] = A_2 - A_1$

$$[\underline{A}] = \underline{A}_2 - \underline{A}_1$$

c. Using similar mathematics, one can obtain

i) $\nabla\cdot\underline{A} \overset{1D}{=} \frac{\partial A_x}{\partial x} \Rightarrow \frac{1}{2}\oint \frac{\partial A_x}{\partial x}\,d\underline{l} = [A_x] = \hat{\underline{x}}\cdot[\underline{A}]$

ii) $\nabla\times\underline{A} = \hat{\underline{z}}\frac{\partial A_y}{\partial x} - \hat{\underline{y}}\frac{\partial A_z}{\partial x} \Rightarrow \frac{1}{2}\oint \nabla\times\underline{A}\cdot d\underline{l} = \hat{\underline{z}}[A_y] - \hat{\underline{y}}[A_z] = \hat{\underline{x}}\times[\underline{A}]$

d. For a general shock normal unit vector $\hat{\underline{n}}$

$$\boxed{\begin{aligned} \nabla A &\rightarrow \hat{\underline{n}}\,[A] \\ \nabla\cdot\underline{A} &\rightarrow \hat{\underline{n}}\cdot[\underline{A}] \\ \nabla\times\underline{A} &\rightarrow \hat{\underline{n}}\times[\underline{A}] \end{aligned}}$$

7. Using this conversion from derivatives to jumps;

a. $\hat{n}\cdot[\rho \mathbf{U}]=0$

b. $\hat{n}\cdot[\rho \mathbf{U}\mathbf{U}]+\hat{n}\left[p+\frac{B^2}{2\mu_0}\right]-\hat{n}\cdot\left[\frac{\mathbf{B}\mathbf{B}}{\mu_0}\right]=0$

c. $\hat{n}\times[\mathbf{U}\times\mathbf{B}]=0$

d. $\hat{n}\cdot[\mathbf{B}]=0$

e. $\hat{n}\cdot\left[\frac{1}{2}\rho U^2\mathbf{U}+\frac{\gamma p}{\gamma-1}\mathbf{U}+\frac{B^2}{\mu_0}\mathbf{U}-\frac{1}{\mu_0}(\mathbf{U}\cdot\mathbf{B})\mathbf{B}\right]=0$

8. We can evaluate the dot & cross products, and split the normal and tangential components of $\mathbf{U}$ & $\mathbf{B}$, to obtain simple jump conditions

a. $\boxed{[\rho U_n]=0}$

b. $[\rho U_n\mathbf{U}]+\hat{n}\left[p+\frac{B^2}{2\mu_0}\right]-\left[\frac{B_n\mathbf{B}}{\mu_0}\right]=0$

i) Take $\mathbf{U}=U_n\hat{n}+\mathbf{U}_t$, $\mathbf{B}=B_n\hat{n}+\mathbf{B}_t$

ii) $\hat{n}$ component:

① $[\rho U_n U_n]+\left[p+\frac{B^2}{2\mu_0}\right]-\cancel{\left[\frac{B_nB_n}{\mu_0}\right]}=0$ $\quad$ 0 since $[B_n]=0$

② $[\rho U_n U_n]=\underbrace{\rho_2U_{n_2}}_{=\rho_1U_{n_1}}U_{n_2}-\rho_1U_{n_1}U_{n_1}=\rho_1U_{n_1}[U_{n_2}-U_{n_1}]=\rho U_n[U_n]$

I. B.8 (Continued)

b. ii) (Continued)

$$\boxed{\rho U_n [U_n] = -\left[p + \frac{B^2}{2\mu_0}\right]}$$

iii) Tangential component:

① $\left[\rho U_n \underset{\sim}{U_t}\right] - \left[\frac{B_n \underset{\sim}{B_t}}{\mu_0}\right] = 0$

② using $[\rho u_n] = 0$ and $[B_n] = 0$,

③ $$\boxed{\rho u_n [\underset{\sim}{u_t}] = \frac{B_n}{\mu_0}[\underset{\sim}{B_t}]}$$

c. $\hat{\underset{\sim}{n}} \times \left[(U_n \hat{\underset{\sim}{n}} + \underset{\sim}{U_t}) \times (B_n \hat{\underset{\sim}{n}} + \underset{\sim}{B_t})\right]$

$= \hat{\underset{\sim}{n}} \times \left[U_n \hat{\underset{\sim}{n}} \times \underset{\sim}{B_t} + B_n \underset{\sim}{U_t} \times \hat{\underset{\sim}{n}} + (\underset{\sim}{U_t} \times \underset{\sim}{B_t})\right]$

i) Easy to show $\hat{\underset{\sim}{n}} \times [\underbrace{\underset{\sim}{U_t} \times \underset{\sim}{B_t}}_{\hat{\underset{\sim}{n}} \text{ direction}}] = 0$!

ii) $\hat{\underset{\sim}{n}} \times (\hat{\underset{\sim}{n}} \times \underset{\sim}{B_t}) = \cancelto{0}{(\hat{\underset{\sim}{n}} \cdot \underset{\sim}{B_t})} \hat{\underset{\sim}{n}} - (\hat{\underset{\sim}{n}} \cdot \hat{\underset{\sim}{n}}) \underset{\sim}{B_t} = -\underset{\sim}{B_t}$

NRL (2) $\underset{\sim}{A} \times (\underset{\sim}{B} \times \underset{\sim}{C}) = (\underset{\sim}{A} \cdot \underset{\sim}{C}) \underset{\sim}{B} - (\underset{\sim}{A} \cdot \underset{\sim}{B}) \underset{\sim}{C}$

iii) Similarly $\hat{\underset{\sim}{n}} \times (\underset{\sim}{U_t} \times \hat{\underset{\sim}{n}}) = \underset{\sim}{U_t}$

iv) Thus $[-U_n \underset{\sim}{B_t} + B_n \underset{\sim}{U_t}] = 0$

v) Using $[B_n] = 0$, $\boxed{B_n [\underset{\sim}{u_t}] = [u_n \underset{\sim}{B_t}]}$

d. $\boxed{[B_n]=0}$

e. $$\left[\frac{1}{2}\rho u^2 u_n + \frac{\gamma p}{\gamma-1}u_n + \frac{B^2}{\mu_0}u_n - \frac{1}{\mu_0}(\mathbf{u}\cdot\mathbf{B})B_n\right]=0$$

i) $$\left[\rho u_n\left(\frac{u^2}{2}+\frac{\gamma p}{(\gamma-1)\rho}\right)+\frac{(\cancel{B_n^2}+|\mathbf{B}_t|^2)u_n}{\mu_0} - \frac{B_n}{\mu_0}(\cancel{u_n B_n}+\mathbf{u}_t\cdot\mathbf{B}_t)\right]=0$$

ii) $$\boxed{\rho u_n\left[\frac{u^2}{2}+\frac{\gamma p}{(\gamma-1)\rho}\right] = \frac{B_n}{\mu_0}\left[\mathbf{u}_t\cdot\mathbf{B}_t\right] - \frac{1}{\mu_0}\left[u_n|\mathbf{B}_t|^2\right]}$$

iii) Note: When $\mathbf{B}=0$, this reduces to the hydrodynamic Rankine-Hugoniot jump condition for energy density (see Lect #15, III.B.3.c.)

iv) Recall specific enthalpy $h \equiv \frac{\gamma p}{(\gamma-1)\rho}$

9. <u>Summary:</u> MHD Rankine-Hugoniot Jump Conditions

a. $[\rho u_n]=0$

b. $[B_n]=0$

c. $\rho u_n[u_n] = -\left[p+\frac{B^2}{2\mu_0}\right]$

d. $\rho u_n[\mathbf{u}_t] = \frac{B_n}{\mu_0}[\mathbf{B}_t]$

e. $B_n[\mathbf{u}_t] = [u_n\mathbf{B}_t]$

f. $\rho u_n\left[\frac{u^2}{2}+\frac{\gamma p}{(\gamma-1)\rho}\right] = \frac{B_n}{\mu_0}[\mathbf{u}_t\cdot\mathbf{B}_t] - \frac{1}{\mu_0}\left[u_n|\mathbf{B}_t|^2\right]$

I. (Continued)

C. Classification of MHD Discontinuities

1. This system of nonlinear jump conditions can be written in a quasi-linear form:

   a. Define: Average Quantity $\langle A \rangle = \frac{1}{2}(A_2 + A_1)$

   b. $[A]\langle B \rangle + \langle A \rangle [B] = (A_2 - A_1)\frac{1}{2}(B_2+B_1) + \frac{1}{2}(A_2+A_1)(B_2-B_1)$

$$= \frac{1}{2}\left\{A_2B_2 + \cancel{A_2B_1} - \cancel{A_1B_2} - A_1B_1 + A_2B_2 - \cancel{A_2B_1} + \cancel{A_1B_2} - A_1B_1\right\}$$

$$= \frac{1}{2}\left\{2A_2B_2 - 2A_1B_1\right\} = A_2B_2 - A_1B_1 = [AB]$$

   c. Thus $\boxed{[AB] = [A]\langle B \rangle + \langle A \rangle [B]}$

2. To create a "linear dispersion relation" like equation for the discontinuities:

   a. Define: $\boxed{F \equiv \rho u_n}$ (Constant) Normal Mass Flux

   b. Define: Specific volume: $\boxed{V \equiv \frac{1}{\rho}}$

   c. Write system of jump conditions in terms of linear jumps and averages

   d. Ex: i) $F[V] - [u_n] = 0$

ii) $\rho u_n \left[\frac{1}{\rho_2} - \frac{1}{\rho_1}\right] - [u_{n2} - u_{n1}] = \left[\frac{\cancel{\rho_2} u_{n2}}{\cancel{\rho_2}} - \frac{\cancel{\rho_1} u_{n1}}{\cancel{\rho_1}}\right] - [u_{n2} - u_{n1}]$

$$= [u_{n2} - u_{n1}] - [u_{n2} - u_{n1}] = 0 \checkmark$$

3. After extensive manipulations, one can obtain a system of jump equations:

a. $F[V] - [u_n] = 0$

b. $F[V] + \hat{n}[p] + \frac{\hat{n}}{\mu_0}\langle \underset{\sim}{B} \rangle \cdot [\underset{\sim}{B}] - \frac{B_n}{\mu_0}[\underset{\sim}{B}] = 0$

c. $F\langle V \rangle [\underset{\sim}{B}] + \langle \underset{\sim}{B} \rangle [u_n] - B_n[\underset{\sim}{u}] = 0$

d. $[B_n] = 0$

e. $F\left\{\frac{\gamma}{\gamma-1}\left(\langle p \rangle [V] + [p]\langle V \rangle\right) + \langle p \rangle [V] + \frac{\langle V \rangle [\underset{\sim}{B}_t]^2}{4\mu_0}\right\} = 0$

f. Unknowns: $\underbrace{[V], [\underset{\sim}{u}], [\underset{\sim}{B}], [p]}_{8 \text{ unknowns}}$ Knowns: $\langle \underset{\sim}{B} \rangle, \langle V \rangle, \langle p \rangle$

g. Equations: 9 → Can eliminate one unknowns → 7 modes.

4. This is a linear system of equations for the jumps.
   a. Determinant must be equal to zero for a solution
   b. Yields condition on F in terms of averages & jumps

5. $$\boxed{F\left(F^2 - \frac{B_n^2}{\mu_0 \langle V \rangle}\right)\left\{F^4 + F^2\left(\frac{[p]}{[V]} - \frac{\langle \underset{\sim}{B} \rangle^2}{\mu_0 \langle V \rangle}\right) - \frac{B_n^2}{\mu_0 \langle V \rangle}\frac{[p]}{[V]}\right\} = 0}$$

Analogous to MHD Linear Wave Dispersion Relation!

I. C (Continued)

6. Solutions: 7th order equation for F

$$\underbrace{F}_{\text{I}}\underbrace{\left(F-\frac{B_n^2}{\mu_0\langle V\rangle}\right)}_{\text{II}}\underbrace{\left\{F^4+F^2\left(\frac{[p]}{[V]}-\frac{\langle \underset{\sim}{B}\rangle^2}{\mu_0\langle V\rangle}\right)-\frac{B_n^2}{\mu_0\langle V\rangle}\frac{[p]}{[V]}\right\}}_{\text{III}}=0$$

7. Class I: $\boxed{F_{I}=0}$ Zero normal mass flux

a. Tangential discontinuity

b. Contact discontinuity

8. Class II: $\boxed{F_{II}=\pm\frac{B_n}{\sqrt{\mu_0\langle V\rangle}}}$

a. Rotational discontinuity

9. Class III: $F_{III}^2=-\frac{1}{2}\left(\frac{[p]}{[V]}-\frac{\langle \underset{\sim}{B}\rangle^2}{\mu_0\langle V\rangle}\right)\pm\sqrt{\frac{\Delta}{4}}$

a. Shocks

b. Discriminant: $\Delta=\left(\frac{[p]}{[V]}-\frac{\langle \underset{\sim}{B}\rangle^2}{\mu_0\langle V\rangle}\right)^2+\frac{4B_n^2}{\mu_0\langle V\rangle}\frac{[p]}{[V]}>0$

c. For a shock to evolve, $F_{III}^2>0$, so negative sign is not realizable

d. Shocks always exist when

$$\frac{\langle \underset{\sim}{B}\rangle^2}{\mu_0\langle V\rangle}>\frac{[p]}{[V]}\Rightarrow$$ Compressive or rarefaction shocks

Next time, we'll explore the properties of these different classes of discontinuities