Lecture #17: MHD Shocks & Discontinuities II

I. Discontinuities & Shocks

Reference: Baumjohann & Treumann, Basic Space Plasma Physics, Imperial College Press, 1997.

A. Classification: Review

1. $$\underbrace{F}_{\mathrm{I}}\underbrace{\left(F^2-\frac{B_n^2}{\mu_0\langle V\rangle}\right)}_{\mathrm{II}}\underbrace{\left\{F^4+F^2\left(\frac{[p]}{[V]}-\frac{\langle \underset{\sim}{B}\rangle^2}{\mu_0\langle V\rangle}\right)-\frac{B_n^2}{\mu_0\langle V\rangle}\frac{[p]}{[V]}\right\}}_{\mathrm{III}}=0$$

a. $F \equiv \rho u_n$ Normal Mass Flux (constant)

b. $V \equiv \frac{1}{\rho}$ specific volume

c. $[A] = A_2 - A_1$ $\langle A\rangle = \frac{1}{2}(A_2 + A_1)$

2. Jump Conditions

(a) $[\rho u_n] = 0$

(b) $[B_n] = 0$

(c) $\rho u_n [u_n] = -\left[p + \frac{B^2}{2\mu_0}\right]$

(d) $\rho u_n [\underset{\sim}{u}_t] = \frac{B_n}{\mu_0}[\underset{\sim}{B}_t]$

(e) $B_n[\underset{\sim}{u}_t] = [u_n \underset{\sim}{B}_t]$

(f) $\rho u_n\left[\frac{u^2}{2} + \frac{\gamma p}{(\gamma-1)\rho}\right] = \frac{B_n}{\mu_0}[\underset{\sim}{u}_t \cdot \underset{\sim}{B}_t] - \frac{1}{\mu_0}[u_n |\underset{\sim}{B}_t|^2]$

B. Class (I): Contact & Tangential discontinuities.

1. $F_I = 0 \Rightarrow u_n = 0$ $\boxed{\text{zero normal mass flux}}$

a. Two Cases: i) $B_n \neq 0$
ii) $B_n = 0$

2. $B_n \neq 0$: Contact Discontinuity

a. Since $u_n = 0$, (d) $\Rightarrow [\underset{\sim}{B}_t] = 0$ (e) $\Rightarrow [\underset{\sim}{u}_t] = 0$

b. (c) $\Rightarrow [p + \frac{B^2}{2\mu_0}] = 0$, but since $[B_n] = 0$ & $[\underset{\sim}{B}_t] = 0$,
$\Rightarrow [p] = 0$

c. Therefore.
$$\boxed{\begin{array}{lll} [p] = 0 & [B_n] = 0 & u_n = 0 \\ [\underset{\sim}{u}_t] = 0 & [\underset{\sim}{B}_t] = 0 & \end{array}}$$

d. The only quantity that can change $\Rightarrow [\rho] \neq 0$. $\boxed{\text{Contact Discontinuity}}$

e. Since $[p] = 0$ and $p = nkT$,
$\boxed{\text{Change in density is balanced by a change in temperature}}$

f. Since $B_n \neq 0$, temperature gradient should be dispersed by parallel electron heat flux (neglected in ideal MHD)
$\Rightarrow$ Contact discontinuities should not persist for long.

3. $B_n = 0$: Tangential Discontinuity

a. Since $B_n = 0$ & $u_n = 0$, (d) & (e) are trivially satisfied

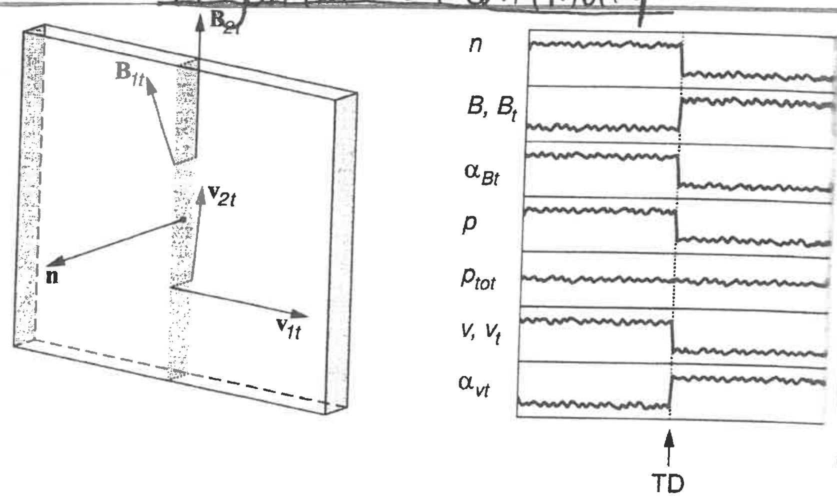
b. (c) $\boxed{[p + \frac{B^2}{2\mu_0}] = 0}$ total pressure balance

c. Plasma flow and magnetic field are tangential to discontinuity $\Rightarrow$ $\boxed{\text{Tangential Discontinuity}}$

d. Tangential magnetic field and velocity may arbitrarily change their magnitudes B_t, v_t and directions α_{Bt}, α_{vt}.

$\Rightarrow$ only total pressure $[p_{tot}] = [p + \frac{B^2}{2\mu_0}] = 0$ is unchanged

Tangential Discontinuity



$\leftarrow$ P_{tot} constant

C. Class (II): Rotational Discontinuity

1. $F_{II} = \pm \dfrac{B_n}{\sqrt{\mu_0 \langle v \rangle}}$

2. For this class, there is no jump in normal velocity $\boxed{[v_n] = 0}$

b. This implies, since $[\rho v_n] = 0$, that $[\rho] = 0$.

$\Rightarrow$ Density is constant $\rightarrow \langle v \rangle = \dfrac{1}{\rho}$

3. $\rho v_n = \pm \dfrac{B_n \sqrt{\rho}}{\sqrt{\mu_0}} \Rightarrow \boxed{v_n = \pm \dfrac{B_n}{\sqrt{\mu_0 \rho}}}$

4. Since $[u_n]=0$, (c) $\Rightarrow$ $\left[p + \frac{B^2}{2\mu_0}\right] = 0$

5. (d) $\underbrace{\rho u_n}_{=\pm B_n\sqrt{\rho/\mu_0}}\,[\underset{\sim}{u}_t] = \frac{B_n}{\mu_0}[B_t]$ $\Rightarrow \pm B_n\sqrt{\frac{\rho}{\mu_0}}\,[\underset{\sim}{u}_t] = \frac{B_n}{\mu_0}[\underset{\sim}{B}_t]$

$$\Rightarrow \boxed{[\underset{\sim}{u}_t] = \frac{[\underset{\sim}{B}_t]}{\sqrt{\mu_0 \rho}}}$$

6. a. (e) $B_n[\underset{\sim}{u}_t] = [u_n \underset{\sim}{B}_t] = \underbrace{\langle u_n\rangle}_{=u_n}[\underset{\sim}{B}_t] + \cancel{[u_n]}\langle \underset{\sim}{B}_t\rangle$

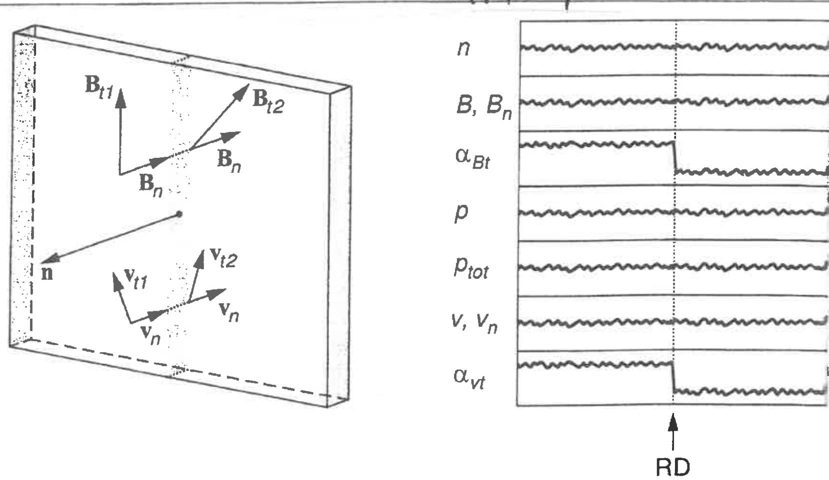
b. $\Rightarrow B_n[\underset{\sim}{u}_t] = u_n[\underset{\sim}{B}_t] \Rightarrow \frac{[\underset{\sim}{u}_t]}{u_n} = \frac{[\underset{\sim}{B}_t]}{B_n}$

c. This relation implies $\underset{\sim}{u}_t$ & $\underset{\sim}{B}_t$ must rotate together across the discontinuity, without changing their magnitudes $\rightarrow [|\underset{\sim}{u}_t|^2]=0$, $[|\underset{\sim}{B}_t|^2]=0$

7. Summary:

$$\boxed{\begin{array}{ll} [\rho]=0 & [B_t^2]=0 \\ [p]=0 & [u_t^2]=0 \\ [u_n]=0 & \\ [B_n]=0 & \left[\underset{\sim}{u}_t - \frac{\underset{\sim}{B}_t}{\sqrt{\mu_0\rho}}\right]=0 \end{array}}$$

Rotational Discontinuity



only magnetic & velocity rotate.

8. "Alfvénic Flow Speed"

Walén relation
Used to determine u_n!

a. Since $\boxed{U_n = \pm \frac{B_n}{\sqrt{\mu_0 \rho}}}$, flow through discontinuity is at the normal component of Alfvén velocity.

b. In a plasma at rest, a rotational discontinuity propagates at the Alfvén speed as a twist in the tangential component of $\underset{\sim}{B}_t$ and $\underset{\sim}{U}_t$.

9. Note: a. Since $[p]=0$ and $[\rho]=0$, $\Rightarrow [T]=0$
$\Rightarrow$ no entropy increase $\longrightarrow$ irreversible

D. Class III: Fast and Slow Magnetosonic Shocks

1. $$F_{III}^4 + F_{III}^2\left(\frac{[p]}{[V]} - \frac{\langle \underset{\sim}{B} \rangle^2}{\mu_0 \langle V \rangle}\right) - \frac{B_n^2}{\mu_0 \langle V \rangle}\frac{[p]}{[V]} = 0$$

2. Intermediate Shock

a. If $[V]=0$ (no density jump), this condition reduces to

$$[p]\left(F_{III}^2 - \frac{B_n^2}{\mu_0 \langle V \rangle}\right) = 0$$

b. Either $[p]=0$ (Continuous pressure)

or $$F_{III} = \pm \frac{B_n}{\sqrt{\mu_0 \langle V \rangle}} \leftarrow$$ same condition as rotational discontinuity, but allows $[p] \neq 0$!

C. <u>Intermediate Shock</u>: $[\rho]\neq 0$ and $u_n = \pm \frac{B_n}{\sqrt{\mu_0 \rho}}$

i) Propagates at Alfvén speed

ii) If $[\rho]=0$, this is just a rotational discontinuity.

$\Rightarrow$ $\boxed{\text{Rotational discontinuities are a sub-class of intermediate shocks with } [\rho]=0 \text{ \& no entropy increase}}$

3. <u>True Shocks</u>:

a. For all other cases, F_{III} has two pairs of conjugate solutions with $[p]\neq 0$ and $[\rho]\neq 0$!

$\Rightarrow$ Thermodynamic state changes across discontinuity.

b. Only the three solutions with $F_{III}^2 > 0$ are relevant.

(i) One solution: $\frac{[p]}{[v]} > 0$

1) For $[p] \geq 0$, this implies $[v] \geq 0$ or $[\rho] < 0$.

2) $\Rightarrow$ <u>Density decreases</u>

3) $\Rightarrow$ $\boxed{\text{Rarefaction Shock}}$

4) Pressure increases, so since $p = nkT$,

$\Rightarrow$ <u>Temperature increases</u>

$\Rightarrow$ Irreversible

(ii) Two solutions: $\frac{[p]}{[v]} < 0$

1) For $[p] > 0$,

$[v] < 0$, or $[\rho] > 0$

$\Rightarrow$ Compressive transitions

2) $\Rightarrow$ $\boxed{\text{Fast and Slow Magnetosonic Shocks}}$

3) NOTE: $[\rho u_n] = 0$

$\Rightarrow$ $[u_n] = -\frac{\langle u_n \rangle}{\langle \rho \rangle}[\rho] \leftarrow$ Flow and density jumps are opposite.

4. Coplanarity Theorem:

a. Use (d) $[\underset{\sim}{U_t}] = \frac{B_n}{\mu_0 \rho U_n}[\underset{\sim}{B_t}]$ to eliminate $[\underset{\sim}{U_t}]$ in (e)

$$B_n\left(\frac{B_n}{\mu_0 \rho U_n}[\underset{\sim}{B_t}]\right) = [U_n \underset{\sim}{B_t}]$$

b. $$\frac{B_n^2}{\mu_0 \rho U_n}[\underset{\sim}{B_t}] = [U_n \underset{\sim}{B_t}]$$

c. Since both sides must be equal (and therefore parallel), their cross product must vanish.

$$\frac{B_n^2}{\mu_0 \rho U_n}[\underset{\sim}{B_t}] \times [U_n \underset{\sim}{B_t}] = 0$$

d. Assuming $B_n \neq 0$ and $U_n \neq 0$, we can resolve [ ]'s,

$$[\underset{\sim}{B_t}] \times [U_n \underset{\sim}{B_t}] = (\underset{\sim}{B_{t2}} - \underset{\sim}{B_{t1}}) \times (U_{n2} \underset{\sim}{B_{t2}} - U_{n1} \underset{\sim}{B_{t1}})$$

$$= -U_{n1} \underset{\sim}{B_{t2}} \times \underset{\sim}{B_{t1}} - U_{n2} \underset{\sim}{B_{t1}} \times \underset{\sim}{B_{t2}} = (U_{n2} - U_{n1})(\underset{\sim}{B_{t2}} \times \underset{\sim}{B_{t1}}) = 0$$

e. $$\boxed{(U_{n2} - U_{n1})(\underset{\sim}{B_{t2}} \times \underset{\sim}{B_{t1}}) = 0}$$

f. Since $[U_n] = 0$ for F_{III} shocks,

$$\boxed{\underset{\sim}{B_{t2}} \times \underset{\sim}{B_{t1}} = 0}$$

Tangential components upstream and downstream are in the same plane!

$\Rightarrow$ $\boxed{\text{Coplanarity}}$

g. Shock has a two-dimensional magnetic geometry!

h. Same condition applies to velocity $\boxed{\underset{\sim}{U}_{t2} \times \underset{\sim}{U}_{t1} = 0}$

i. Two dimensions of $\underset{\sim}{U}_t$ and $\hat{n}$ may be different from those of $\underset{\sim}{B}_t$ and $\hat{\underset{\sim}{n}}$.

5. Jump Conditions for F_{III} Shocks

a. Combining (f) $\rho u_n \left[\frac{U^2}{2} + \frac{\gamma p}{(\gamma-1)\rho}\right] = \frac{B_n}{\mu_0}[\underset{\sim}{U}_t \cdot \underset{\sim}{B}_t] - \frac{1}{\mu_0}[u_n |\underset{\sim}{B}_t|^2]$

with (d) $[\underset{\sim}{U}_t] = \frac{B_n}{\mu_0 F_{III}}[\underset{\sim}{B}_t]$

and splitting products of jumps, we can obtain

$$\boxed{[p]\langle u_n\rangle + [u_n]\gamma\langle p\rangle + \frac{\gamma-1}{4\mu_0}[u_n][B_t^2] = 0}$$

b. Due to coplanarity theorem, we can write the combination of (d) & (e) in I.D.4.b as a scalar equation

$$\boxed{[u_n]\langle B_t\rangle + [B_t]\langle u_n\rangle = \frac{B_n^2}{\mu_0 F_{III}}[B_t]}$$

c. (d) $\Rightarrow$ $\boxed{B_n[U_t] = \frac{B_n^2}{\mu_0 \rho}[B_t]}$

d. (c) $\Rightarrow$ $\boxed{F_{III}[u_n] = -\left[p + \frac{B^2}{2\mu_0}\right]}$

These four boxed equations are a closed set for $[p]$, $[u_n]$, $[U_t]$, & $[B_t]$ and are the general Rankine-Hugoniot conditions for MHD Shocks

6. Fast and Slow Magnetosonic Shocks

a. Eliminating $[u_n]$ from those conditions yields

$$\left(\frac{\langle u_n\rangle}{\gamma-1} - H\right)[p] = \frac{H}{\mu_0 F_{III}}[B_t^2]$$

where $F_{III} H = \frac{[B_t^2]}{4\mu_0} + \frac{\gamma\langle p\rangle}{\gamma-1}$

b. For a shock, pressure always increases $[p]>0$.

c. Two types of shocks:

i) For $[B_t^2]>0$, $\langle u_n\rangle > (\gamma-1)H$ $\boxed{\text{Fast Shock}}$

ii) For $[B_t^2]<0$, $\langle u_n\rangle < (\gamma-1)H$ $\boxed{\text{Slow Shock}}$

d. Slow shocks only exist when $\langle p\rangle$ is large enough to satisfy

$$\gamma\langle p\rangle - \frac{\gamma-1}{4\mu_0}\left(B_{t_1}B_{t_2} - [B_t^2]\right) > 0$$

→ needed so $u_n>0$ (flow from undisturbed to disturbed)

I. D. 6. (Continued)

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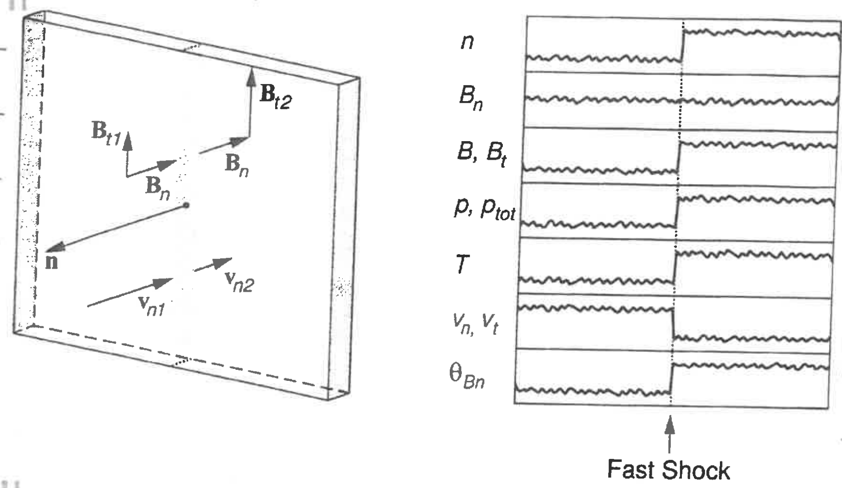
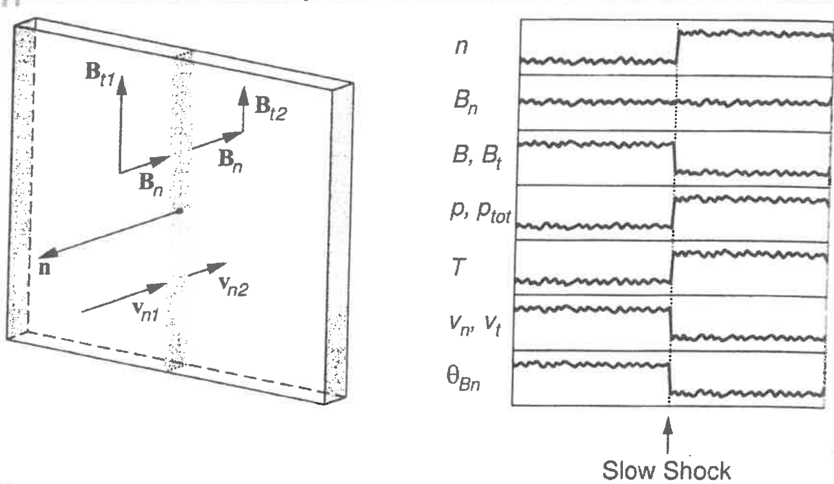
e. Fast Shock



← $|\underset{\sim}{B}|$ increases

← $\underset{\sim}{B}$ bends away from shock normal

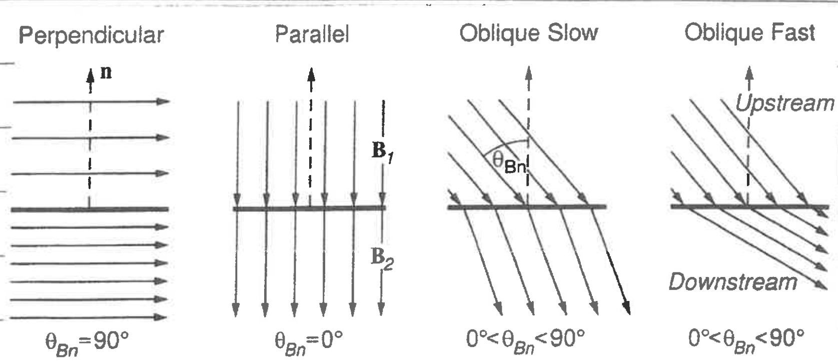
f. Slow Shock



← $|\underset{\sim}{B}|$ decreases

← $\underset{\sim}{B}$ bends towards shock normal

g. Magnetic Field Geometry



E. Entropy Changes

1. Define: Entropy $\boxed{S \equiv \frac{p}{\rho^\gamma}}$

2. The adiabatic equation of state, which gives $\frac{\partial p}{\partial t}$, can be written

$$\boxed{\frac{ds}{dt} = 0} \qquad \text{where } \frac{d}{dt} = \frac{\partial}{\partial t} + \underset{\sim}{U} \cdot \nabla$$

Lagrangian derivative

3. Thus

$$\cancel{\frac{\partial s}{\partial t}}^{\,0 \text{ steady state.}} + \underset{\sim}{U} \cdot \nabla S = 0$$

a. $\underset{\sim}{U} \cdot \nabla S = 0 \quad \Rightarrow \quad \nabla \cdot (S \underset{\sim}{U}) = S\, \nabla \cdot \underset{\sim}{U} + \underset{\sim}{U} \cdot \nabla S$

$$\Rightarrow \nabla \cdot (S \underset{\sim}{U}) = S\, \nabla \cdot \underset{\sim}{U}$$

b. Performing the line integral across the shock & back (1D)

$$[S U_n] = \int dn\, s \frac{dU_n}{dn} = \int s\, dU_n$$

c. If $[U_n] = 0$, then $[s] = 0 \rightarrow$ isentropic

d. $\boxed{\text{When } [U_n] \neq 0 \text{, the jump in } U_n \text{ will lead to an increase in entropy across the shock}}$

Thus, irreversible changes across shock.