Hawes (1)

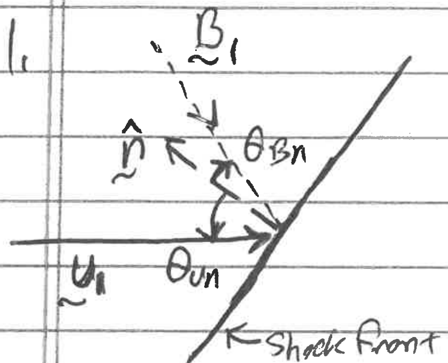
Lecture #18

Ref: Burgess & Scholer, Collisionless Shocks in Space Plasmas, Cambridge (2015)

I. MHD Shock Geometry:

A. Earth's Bow Shock

1.



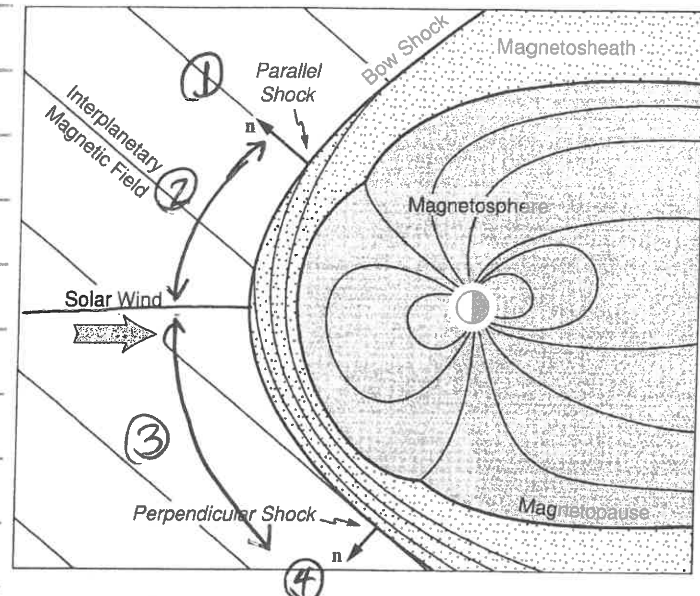


Fig. 8.10. Parallel and perpendicular bow shock regions.

2. Def: Shock Normal

$\underset{\sim}{\hat{n}}$ – direction perpendicular to shock surface.

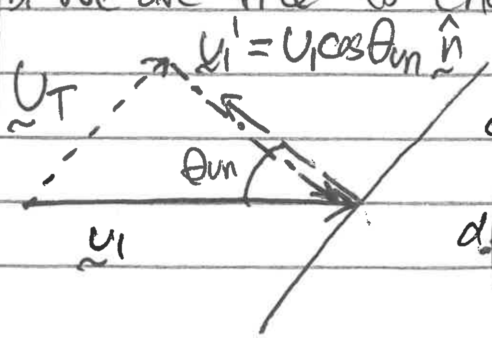
3. a. Def: Shock normal angle of magnetic field, θ_{Bn}

b. Def: Shock normal angle of upstream flow, θ_{Un}

4. Def: Normal Incidence Frame (NIF)

a. Frame of reference in which upstream flow is parallel to the normal.

b. We are free to change to NIF frame.



c. $\underset{\sim}{U}_1' = \underset{\sim}{U}_1 + \underset{\sim}{U}_T = U_1 \cos\theta_{Un} \hat{\underset{\sim}{n}}$

d. $\boxed{\underset{\sim}{U}_T = \underset{\sim}{U}_1 - U_1 \cos\theta_{Un} \hat{\underset{\sim}{n}}}$

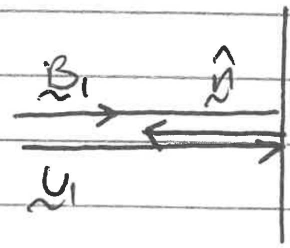
$\Big\}\ \underset{\sim}{U}_1' \parallel \hat{\underset{\sim}{n}}$

II.A.4. (Continued)

e. In the NIF, θ_{Bn} is eliminated as a parameter.

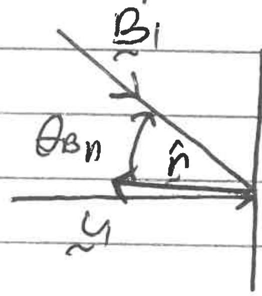
B. <u>Shock-Normal Angle Ranges:</u> (NIF frame)

(1) <u>Parallel Shock:</u> $\theta_{Bn} = 0^\circ$

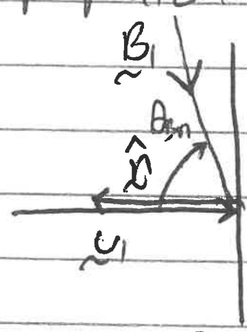


a. $\underset{\sim}{U}_1 \parallel \underset{\sim}{B}_1 \parallel \hat{n}$

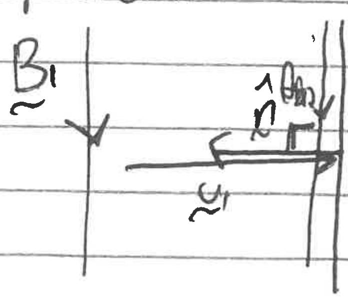
(2) Quasi-parallel Shock $0^\circ < \theta_{Bn} \leq 45^\circ$



(3) Quasi-perpendicular Shock $45^\circ \leq \theta_{Bn} < 90^\circ$



(4) Perpendicular Shock $\theta_{Bn} = 90^\circ$



B. MHD Shock Parameters

1. For a stationary Bow shock at the Earth, the shock will be determined by the upstream flow parameters.

subscript "1" – upstream
"2" – downstream

$$p_1 \, , \, \rho_1 \, , \, \underset{\sim}{B}_1 \, , \, \underset{\sim}{u}_1$$

2. Transforming frame to NIF $\underset{\sim}{u}_1 \rightarrow u_n$ (eliminate θ_{un})

3. Due to coplanarity theorem $\underset{\sim}{B}_1$ and $\underset{\sim}{B}_2$ lie in the same plane (for compressive shocks), so we can use $B_1 = |\underset{\sim}{B}_1|$ and θ_{Bn} to parameterize.

4. In the NIF, MHD shocks can be characterized by three dimensionless parameters.

a. Upstream Alfvén Mach number: $\boxed{M_A \equiv \frac{u_1}{V_{A1}}}$

where $V_{A1} = \frac{B_1}{\sqrt{\mu_0 \rho_1}}$

b. Upstream Plasma Beta: $\boxed{\beta_1 = \frac{2\mu_0 n_1 k T_1}{B_1^2} = \frac{2\mu_0 p_1}{B_1^2}}$

(Ratio of thermal pressure $p_1 = n_1 k T_1$ to mag press $p_{B_1} = \frac{B_1^2}{2\mu_0}$)

c. Shock-normal angle $\boxed{\theta_{Bn}}$

5. Recall, for linear MHD waves, $\omega = \omega(\beta, \theta_{Bk})$

C. Solutions of MHD Rankine-Hugoniot Jump Conditions

1. Def: Shock Compression Ratio $\boxed{r \equiv \frac{\rho_2}{\rho_1}}$

2. We can define all of the jumps in terms of $M_A, \beta_1, \theta_{Bn}$
   a. Below, we assume monatomic plasma, $\gamma = 5/3$

3. Compression, r:

(A)
$$\boxed{\begin{aligned}&\cos^2\theta_{Bn}\left(2M_A^2 + 5\beta_1\cos^2\theta_{Bn}\right) r^3 \\ &+ M_A^2\left\{M_A^2 - \cos^2\theta_{Bn}\left(5M_A^2 + 8 + 10\beta_1\right)\right\} r^2 \\ &+ M_A^4\left(11\cos^2\theta_{Bn} + 2M_A^2 + 5 + 5\beta_1\right) r - 8M_A^6 = 0\end{aligned}}$$

4. Tranverse Magnetic Field, B_{t2}

(B)
$$\boxed{\frac{B_{t_2}}{B_{t_1}} = \frac{r\left(M_A^2 - \cos^2\theta_{Bn}\right)}{\left(M_A^2 - r\cos^2\theta_{Bn}\right)}}$$

5. Transverse Velocity, U_{t2}

(C)
$$\boxed{\frac{U_{t2}}{U_1} = \frac{\sin\theta_{Bn}\cos\theta_{Bn}(r-1)}{\left(M_A^2 - r\cos^2\theta_{Bn}\right)}}$$

6. Downstream Pressure p_2

(D)
$$\boxed{\frac{p_2}{\left(\rho_1 U_1^2/2\right)} = \frac{1}{2}\left\{\frac{\beta_1}{M_A^2} + \frac{2(r-1)}{r} + \frac{\sin^2\theta_{Bn}}{M_A^2}\left[1 - \left(\frac{B_{t2}}{B_{t1}}\right)^2\right]\right\}}$$

7. **NOTE:** $p = nkT \Rightarrow p = \rho \frac{kT}{m} \Rightarrow \boxed{\frac{kT}{m} = \frac{p}{\rho}}$

a. Can be used to convert p_2 to T_2 given ρ_2.

8. Coplanarity and downstream tangential velocity

a. In the NIF, upstream $\boxed{U_{t_1} = 0}$!

b. Magnetic tension (due to bending of $\underline{B}$ at the shock discontinuity) can impart an impulse that generates $U_{t_2} \neq 0$. This is distinct from hydrodynamic shocks.

c. There is no magnetic tension force perpendicular to the magnetic coplanarity plane.

d. Thus, $\underset{\sim}{U}_{t_2} \neq 0$ must lie in the same coplanarity plane containing $\underset{\sim}{B}_{t_1}$ and $\underset{\sim}{B}_{t_2}$!

e. NOTE: Within the shock transition, there may be components of $\underset{\sim}{U}_t$ and $\underset{\sim}{B}_t$ not within the coplanarity plane. But, downstream $\underset{\sim}{U}_{t_2}$ and $\underset{\sim}{B}_{t_2}$ will lie in the coplanarity plane.

D. Strong Shock Limit: $M_A \gg 1$

1. In this limit $M_A \gg 1$, (A) reduces to

$$2r\cancel{M_A^6} - 8\cancel{M_A^6} = 0 \Rightarrow \boxed{r = \frac{\rho_2}{\rho_1} = 4}$$

I. D. (Continued) Howes ⑥

2. Transverse magnetic field jump (B): a. $\frac{B_{t2}}{B_{t1}} = \frac{r\cancel{(M_A^2 - \cos^2\theta_{Bn})}}{\cancel{(M_A^2 - r\cos^2\theta_{Bn})}}$

b. $\boxed{\frac{B_{t2}}{B_{t1}} = r = 4}$

3. Pressure (p): a. $\frac{p_2}{(\frac{1}{2}\rho_1 u_1^2)} = \frac{1}{\cancel{2}} \frac{\cancel{2}(r-1)}{r} \simeq \frac{4-1}{4} = \frac{3}{4}$

b. Thus $\boxed{p_2 \simeq \frac{3}{4}(\frac{1}{2}\rho_1 u_1^2)}$ Downstream pressure jump is unlimited

c. NOTE: Pressure and energy density have the same units

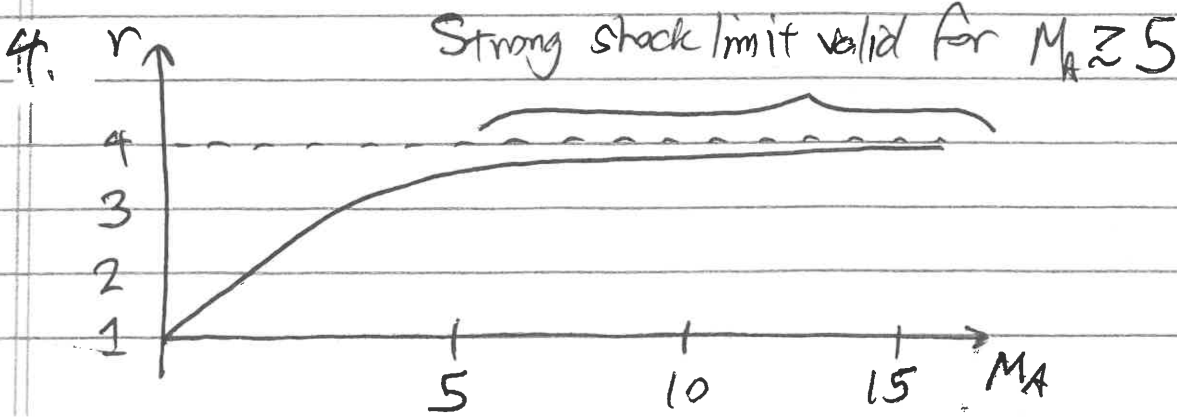
$[p] = \frac{N}{m^2} \qquad [\frac{1}{2}\rho_1 u_1^2] = \frac{J}{m^3} = \frac{N \cdot \cancel{m}}{m^{\cancel{3}2}} = \frac{N}{m^2}$ ✓

d. For a strong shock with $U_2 \to \frac{1}{4}U_1$, (since $r = \frac{\rho_2}{\rho_1} = \frac{U_1}{U_2}$)
what is the change in kinetic energy of the flow?

i) $\frac{1}{2}\rho_1 U_1^2 - \frac{1}{2}\rho_2 U_2^2 = \frac{1}{2}\underbrace{\rho U}_{\text{constant}}(U_1 - U_2) = \frac{1}{2}\rho U(U_1 - \frac{U_1}{4}) = \frac{3}{4}(\frac{1}{2}\rho_1 U_1^2)$

ii) Thus, the lose incoming kinetic energy of the flow is converted to thermal energy!

Upstream kinetic energy ⇒ downstream thermal energy!

4. 

E. <u>Perpendicular Shock Limit</u>: $\theta_{Bn} = 90°$

1. Setting $\cos\theta_{Bn} = 0$, Ⓐ reduces to a quadratic

a. $r^2 + 2r\left[M_A^2 + \frac{5}{2}(1+\beta_1)\right] - 8 M_A^2 = 0$

b. $r = -\left[M_A^2 + \frac{5}{2}(1+\beta_1)\right] \pm \sqrt{\left[M_A^2 + \frac{5}{2}(1+\beta_1)\right]^2 + 8 M_A^2}$

Since $r > 0$ for a shock, only take ⊕ sign!

2. <u>Alfven Mach number threshold for a perpendicular shock</u>:

a. A shock occurs when $r > 1$, so one may set $r = 1$ and solve for M_A to determine the threshold for a shock.

b. $1 = -\underbrace{\left[M_A^2 + \frac{5}{2}(1+\beta_1)\right]}_{=A} + \sqrt{\underbrace{\left[M_A^2 + \frac{5}{2}(1+\beta_1)\right]}_{=A}^2 + 8 M_A^2}$

c. $(1 + A)^2 = A^2 + 8 M_A^2$

$1 + 2A + \cancel{A^2} = \cancel{A^2} + 8 M_A^2$

$1 + 2 M_A^2 + 5(1+\beta_1) = 8 M_A^2$

$6 + 5\beta_1 = 6 M_A^2$

$1 + \frac{5}{6}\beta_1 = M_A^2$

d. <u>Note!</u> $M_A^2 = \frac{U_1^2}{V_A^2}$, so $U_1^2 = V_A^2 + \frac{5}{6}\beta_1 V_A^2$

i) $\frac{5}{6}\beta_1 V_A^2 = \frac{5}{6}\,\frac{2\cancel{\mu_0} p_1}{\cancel{B_1^2}}\,\frac{\cancel{B_1^2}}{\cancel{\mu_0}\rho_1} = \frac{5}{3}\frac{p_1}{\rho_1} = \boxed{\frac{\gamma p_1}{\rho_1} = C_s^2}$

ii) Thus $U_1^2 = V_A^2 + C_s^2$

e. Thus, threshold is $\frac{U_1^2}{V_A^2+C_s^2}=1$

f. For $\theta_{Bn}=90°$, the fast magnetosonic speed is

$$V_f^2 = V_A^2 + C_s^2 \quad \text{for } \theta_{Bn}=90°$$

g. <u>Define:</u> Fast Magnetosonic Mach Number, (Perpendicular shock) $\boxed{M_f \equiv \frac{U_1}{V_f}}$

h. Fast Magnetosonic Shock occurs when $\boxed{M_f > 1}$

F. <u>Parallel Shock Limit:</u> $\theta_{Bn}=0°$

1. Setting $\cos\theta_{Bn}=1$, (A) can be factored to obtain three modes for r.

2. The magnetosonic mode (which depends on β_1) yields a threshold upstream flow of

$$U_1 = C_s$$

a. <u>Define:</u> Sonic Mach Number $\boxed{M \equiv \frac{U_1}{C_s}}$

where $C_s^2 = \frac{\gamma p_1}{\rho_1}$

b. Parallel Shock occurs when $\boxed{M > 1}$

c. This is equivalent to a hydrodynamic shock (See Lec #15).

i) Magnetic field drops out

ii) Parallel MHD Shock is purely hydrodynamic.

G. Oblique Shocks: $0^\circ < \theta_{Bn} < 90^\circ$

1. For oblique shocks, (A) has three different roots for r:
   a. Fast magnetosonic Shock
   b. Slow magnetosonic Shock
   c. Intermediate (Alfvénic) Shock

2. To determine the threshold inflow speed for each root, set $r=1$, and (A) can be factored to

$$\underbrace{\left(M_A^2-\cos^2\theta_{Bn}\right)}_{\text{Intermediate Shock}}\underbrace{\left\{M_A^4-M_A^2\left(1+\tfrac{5}{6}\beta_1\right)+\tfrac{5}{6}\beta_1\cos^2\theta_{Bn}\right\}}_{\text{Fast \& Slow Shocks}}=0$$

a. MHD Wave Speeds:

$$\left.\begin{matrix} V_f \\ V_s \end{matrix}\right\} = \left\{\frac{1}{2}\left[V_A^2+C_s^2 \underset{\text{Slow}}{\overset{\text{Fast}}{\pm}} \sqrt{(V_A^2+C_s^2)^2-4V_A^2C_s^2\cos^2\theta_{Bn}}\right]\right\}^{\frac{1}{2}}$$

Group velocity of Alfven wave along $\hat{n}$ → $V_{gA} = V_A\cos\theta_{Bn}$

Alfven speed (constant) $\boxed{V_A^2=\frac{B_1^2}{\mu_0\rho_1}}$

Sound speed $\boxed{C_s^2=\frac{\gamma p_1}{\rho_1}}$

b. $M_f = \frac{U_1}{V_f}$

Formally, V_f & V_s depend on θ_{Bn} (and β_1)

c. $M_s = \frac{U_1}{V_s}$

d. $M_A = \frac{U_1}{V_A}$

NOTE: Since it is difficult observationally to determine θ_{Bn}, M_A is defined relative to the (total) Alfven velocity.

3. Slow vs. Fast Magnetosonic Shocks:

a. A slow shock will form if $M_s > 1$ but $M_f < 1$

b. This is a narrow range of upstream velocities.

c. For heliospheric shocks, typically $M_f > 1$, so we usually see only fast magnetosonic shocks.

e.g. Earth's bow shock: $M_A \sim 10 \Rightarrow M_f \gtrsim 3$ for typically β_1 values.

Reference:

D. Burgess & M. Scholer, "Collisionless Shocks in Space Plasmas: Structure and Accelerated Particles", Cambridge University Press, 2015.