Hawes ①

Lecture #19: Shock Upstream Conditions, deHoffmann-Teller Frame, and Supercriticality

I. Shock Set up & Reference Frames

A. Jump Conditions for Electric Field

1. In the standard MHD formulation, $\underset{\sim}{E}$ can be eliminated using Ohm's Law (e.g. ideal Ohm's Law, $\underset{\sim}{E} = -\underset{\sim}{U} \times \underset{\sim}{B}$)

2. But, in collisionless plasmas, it is helpful to explicitly consider the electric field to understand particle acceleration.

3. In the shock rest frame, Faraday's Law reduces to

a. $$\cancel{-\frac{\partial \underset{\sim}{B}}{\partial t}}^{\,0} = +\nabla \times \underset{\sim}{E} \quad \Rightarrow \quad \nabla \times \underset{\sim}{E} = 0$$

b. Integrating over a closed line integral from upstream to downstream and back (see Lec #16. I. B. 6)

$$\nabla \times \underset{\sim}{E} = 0 \longrightarrow \hat{\underset{\sim}{n}} \times [\underset{\sim}{E}] = 0$$

c. $$[\hat{\underset{\sim}{n}} \times (\cancel{E_n \hat{\underset{\sim}{n}}}^{\,0} + \underset{\sim}{E}_t)] = 0 \Rightarrow \hat{\underset{\sim}{n}} \times [\underset{\sim}{E}_t] = 0 \Rightarrow \boxed{[\underset{\sim}{E}_t] = 0}$$

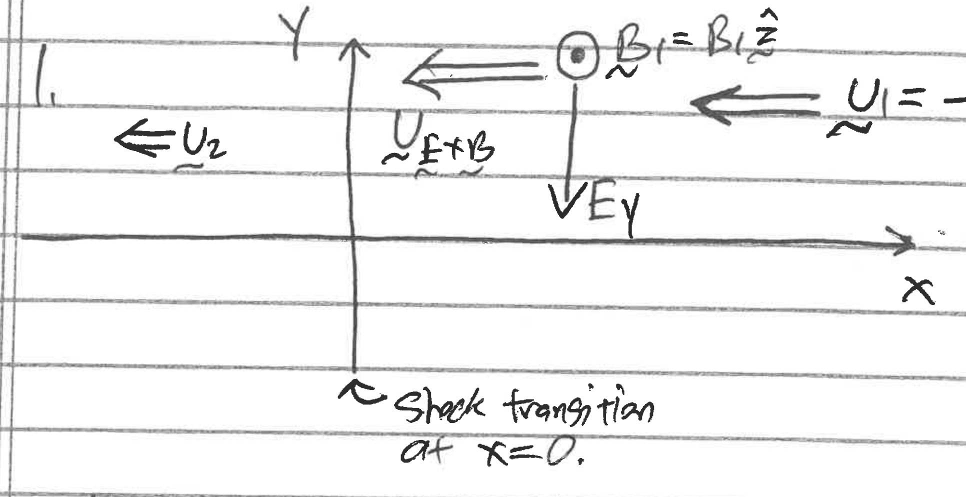
4. Similarly, recall $\boxed{[B_n] = 0}$

5. Thus, Tangential component $\underset{\sim}{E}_t$ is constant and normal component B_n is constant.

B. Set up for a Perpendicular Shock

"1" – upstream
"2" – downstream

1.



where $U_1 > 0$
$B_1 > 0$

2. DEF: Motional Electric Field $\underset{\sim}{E}_1$

a. Since the magnetized plasma is flowing (in the $-\hat{x}$ direction) in the shock-rest frame, there must be a convection electric field.

b. Ideal Ohm's Law: $\underset{\sim}{E} = -\underset{\sim}{U} \times \underset{\sim}{B}$

c. $\underset{\sim}{E} = -\left[(-U_1 \hat{x}) \times (B_1 \hat{z})\right] = -\left[+U_1 B_1 \hat{y}\right] = -U_1 B_1 \hat{y}$

d. Thus $\boxed{E_y = -U_1 B_1 < 0}$ is the motional electric field

e. NOTE: Upstream inflow is simply the $\underset{\sim}{E} \times \underset{\sim}{B}$ velocity

i) $\underset{\sim}{U}_{E\times B} = \dfrac{\underset{\sim}{E} \times \underset{\sim}{B}}{B^2} = \dfrac{(-U_1 B_1 \hat{y}) \times B_1 \hat{z}}{B_1^2} = -U_1 \hat{x}$ ✓

3. NOTE: a. For a perpendicular ($\theta_{Bn} = 90°$) shock with compression ratio $r = \frac{\rho_2}{\rho_1}$ we also have

$[\rho U] = 0 \Rightarrow \dfrac{U_1}{U_2} = r$, $\quad \dfrac{B_{t2}}{B_{t1}} = r$, $\quad B_n = \text{const} = 0$, $\quad E_y = \text{const}$

I. B. 3. (Continued)

b. Downstream, due to coplanarity & $B_n = 0$, $\underset{\sim}{B_2} = B_2 \hat{z}$.

i) Downstream $\underset{\sim}{U}_{\underset{\sim}{E}\times\underset{\sim}{B}\,2} = \dfrac{\underset{\sim}{E_2} \times \underset{\sim}{B_2}}{B_2^2} = \dfrac{(-U_1 B_1 \hat{y}) \times (r B_1 \hat{z})}{(r B_1)^2}$

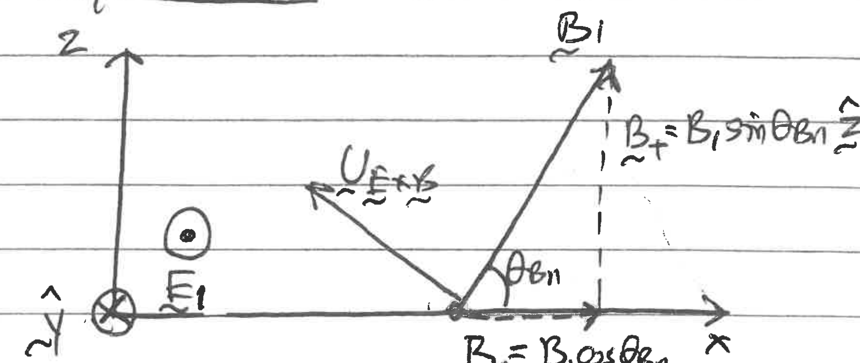
$\Rightarrow \underset{\sim}{E_2} = \underset{\sim}{E_1} = -U_1 B_1 \hat{y}$ $\qquad = \dfrac{-U_1 \cancel{B_1^2}\, \cancel{r}}{r^{\cancel{2}} \cancel{B_1^2}} \hat{x} = -\dfrac{U_1}{r} \underset{\sim}{\hat{x}}$

$B_2 = r B_1$

ii) Check: $\underset{\sim}{U_2} = -\dfrac{U_1}{r} \underset{\sim}{\hat{x}}$ $\qquad \dfrac{U_2}{U_1} = \dfrac{\left(\frac{\cancel{U_1}}{r}\right)}{\cancel{U_1}} \Rightarrow \dfrac{U_1}{U_2} = r$ ✓

C. Oblique Shock: $0^\circ < \theta_{Bn} < 90^\circ$

1.



a. $\underset{\sim}{B_1} = B_1 \cos\theta_{Bn}\, \underset{\sim}{\hat{x}} + B_1 \sin\theta_{Bn}\, \underset{\sim}{\hat{z}}$

b. $\underset{\sim}{E_1} = E_{y1} \underset{\sim}{\hat{y}}$ where $E_{y1} < 0$

2. $\underset{\sim}{E} \times \underset{\sim}{B}$ velocity: a) $\underset{\sim}{U}_{\underset{\sim}{E}\times\underset{\sim}{B}} = \dfrac{(E_{y1} \hat{y}) \times [\cancel{B_1} \cos\theta_{Bn}\, \hat{x} + \cancel{B_1} \sin\theta_{Bn}\, \hat{z}]}{B_1^{\cancel{2}}}$

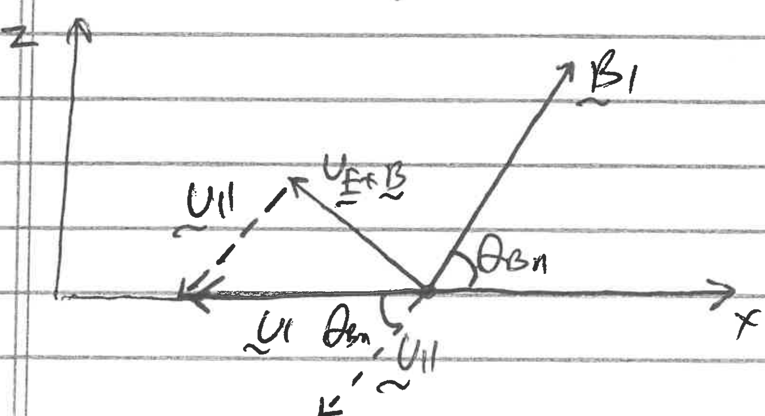
$$\boxed{\underset{\sim}{U}_{\underset{\sim}{E}\times\underset{\sim}{B}} = \underbrace{\left(\frac{-E_{y1}}{B_1}\right)}_{\text{positive}} \left(\cos\theta_{Bn}\, \underset{\sim}{\hat{z}} - \sin\theta_{Bn}\, \underset{\sim}{\hat{x}}\right)}$$

3. Normal Incidence Frame (NIF):

a. To connect with our previous study of the MHD Rankine-Hugoniot conditions, we want $\underline{U}_1 \parallel \hat{\underline{n}} = \hat{\underline{x}}$.

b. We are only free to add a flow along the magnetic field $\underline{U}_\parallel$, so we add a $\underline{V}_\parallel$ such that $(\underline{U}_\parallel + \underline{U}_{E\times B})$ is in the $\hat{\underline{n}} = \hat{\underline{x}}$ direction.

c.



d. $\underline{U}_\parallel = -U_\parallel \cos\theta_{Bn}\hat{\underline{x}} - U_\parallel \sin\theta_{Bn}\hat{\underline{z}}$ where $U_\parallel > 0$

e. $$\underline{U}_1 = \underline{U}_{E\times B} + \underline{U}_\parallel = \left(\frac{-E_{y1}}{B_1}\right)\left(\cos\theta_{Bn}\hat{\underline{z}} - \sin\theta_{Bn}\hat{\underline{x}}\right) + \left(-U_\parallel\cos\theta_{Bn}\hat{\underline{x}} - U_\parallel\sin\theta_{Bn}\hat{\underline{z}}\right)$$

$$= -\left[\left(\frac{-E_{y1}}{B_1}\right)\sin\theta_{Bn} + U_\parallel\cos\theta_{Bn}\right]\hat{\underline{x}} + \left[\left(\frac{-E_{y1}}{B_1}\right)\cos\theta_{Bn} - U_\parallel\sin\theta_{Bn}\right]\hat{\underline{z}}$$

f. We want $\underline{U}_1 \cdot \hat{\underline{z}} = 0$, so

$$\boxed{\frac{-E_{y1}}{B_1}\cos\theta_{Bn} = U_\parallel \sin\theta_{Bn}}$$

g. Inflow velocity $\boxed{U_1 = \underline{U}_1 \cdot \hat{\underline{x}} = -\left[\left(\frac{-E_{y1}}{B_1}\right)\sin\theta_{Bn} + U_\parallel\cos\theta_{Bn}\right]}$

4. Write expressions for E_{y1} and $u_{\parallel}$ in terms of M_A and θ_{Bn}:

a. $M_A = \frac{|U_1|}{V_A} = \left(\frac{-E_{y1}}{B_1 V_A}\right)\sin\theta_{Bn} + \frac{U_{\parallel}}{V_A}\cos\theta_{Bn}$

b. $\frac{U_{\parallel}}{V_A} = \left(\frac{-E_{y1}}{B_1 V_A}\right)\frac{\cos\theta_{Bn}}{\sin\theta_{Bn}}$

c. $M_A = \left(\frac{-E_{y1}}{B_1 V_A}\right)\left(\sin\theta_{Bn} + \frac{\cos^2\theta_{Bn}}{\sin\theta_{Bn}}\right) = \frac{1}{\sin\theta_{Bn}}\left(\frac{-E_{y1}}{B_1 V_A}\right)\cancelto{1}{(\sin^2\theta_{Bn}+\cos^2\theta_{Bn})}$

d. $\boxed{\frac{E_{y1}}{B_1 V_A} = -M_A \sin\theta_{Bn}}$

e. $\boxed{\frac{U_{\parallel}}{V_A} = M_A \cos\theta_{Bn}}$

5. In Summary,

a. $\underset{\sim}{U_1} = \underset{\sim}{U}_{E\times B} + \underset{\sim}{U_{\parallel}} = -M_A V_A \hat{x}$

b. $\underset{\sim}{U_{\parallel}} = -M_A V_A \cos\theta_{Bn}(\cos\theta_{Bn}\,\hat{x} + \sin\theta_{Bn}\,\hat{z})$

c. $\underset{\sim}{E_1} = -M_A V_A B_1 \sin\theta_{Bn}\,\hat{y}$

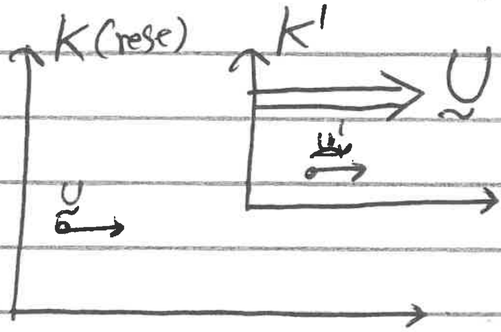
d. $\underset{\sim}{B_1} = B_1\cos\theta_{Bn}\,\hat{x} + B_1\sin\theta_{Bn}\,\hat{z}$

e. $\underset{\sim}{U}_{E\times B} = \frac{\underset{\sim}{E_1}\times\underset{\sim}{B_1}}{B_1^2}$

D. Lorentz Transformation of Electromagnetic Fields

Ref: Appendix of Howes, Klein, & TenBarge, ApJ 789:106 (2014)

1.



a. Frame K at rest

b. Frame K' moving at velocity U relative to K

c. $u = u' + U$ or $u' = u - U$

2. For non-relativistic conditions $\frac{U}{c} \ll 1$,

Fields in frame K' → $\boxed{E' = E + U \times B}$ ← Fields in K

$\boxed{B' = B}$

Frame Transformation velocity

E. de Hoffmann–Teller Frame (HTF)

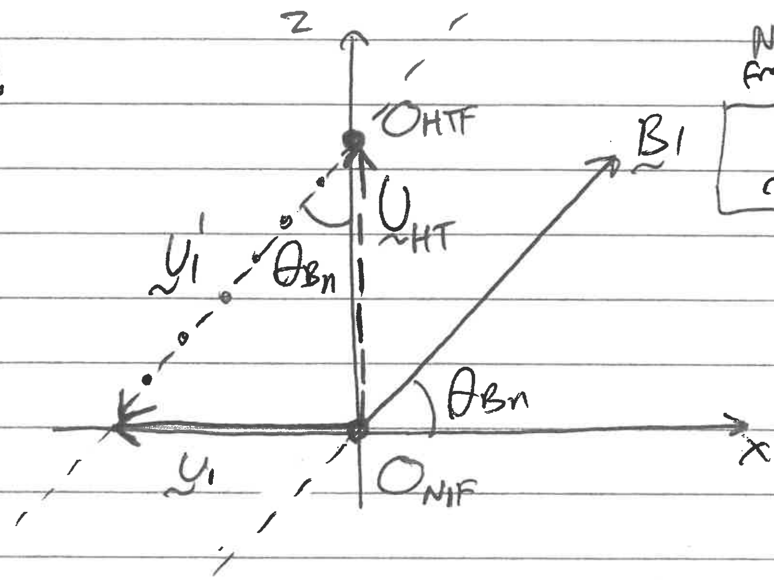
Ref: de Hoffmann & Teller, Phys. Rev. 80:692 (1950)

1. Although the Normal Incidence Frame (NIF) is very useful, the presence of a non-zero upstream motional electric field leads to particle motion involving complicated $E \times B$ drift trajectories.
2. By a useful frame transformation, it is possible to align the upstream flow to the magnetic field, thereby eliminating the upstream motional electric field.

⇒ $\boxed{\text{de Hoffmann–Teller Frame (HTF)}}$

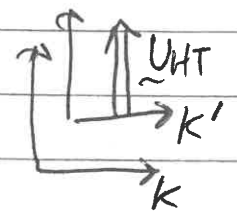
3. We still want a frame in which the shock front is at rest (shock rest frame) so that the shock is stationary ($\frac{\partial}{\partial t} = 0$). We can add <u>any</u> tangential flow to a shock rest frame.

4.



NIF frame, HTF Frame, Transformation velocity

$$\boxed{\underset{\sim}{u}_1 = \underset{\sim}{u}_1' + \underset{\sim}{U}_{HT}}$$



a. $\underset{\sim}{U}_{HT} = \underset{\sim}{u}_1 - \underset{\sim}{u}_1' = u_1 \tan\theta_{Bn}\,\hat{\underset{\sim}{z}} = M_A V_A \tan\theta_{Bn}\,\hat{\underset{\sim}{z}}$

5. Determine $\underset{\sim}{E}_1' = \underset{\sim}{E}_1 + \underset{\sim}{U}_{HT} \times \underset{\sim}{B}_1$

$$= -M_A V_A B_1 \sin\theta_{Bn}\,\hat{\underset{\sim}{y}} + \left[M_A V_A \tan\theta_{Bn}\,\hat{\underset{\sim}{z}}\right] \times \left(B_1 \cos\theta_{Bn}\,\hat{\underset{\sim}{x}} + \cancel{B_1 \sin\theta_{Bn}\,\hat{\underset{\sim}{z}}}\right)$$

$$= -M_A V_A B_1 \sin\theta_{Bn}\,\hat{\underset{\sim}{y}} + M_A V_A \frac{\sin\theta_{Bn}}{\cancel{\cos\theta_{Bn}}} B_1 \cancel{\cos\theta_{Bn}}\,\hat{\underset{\sim}{y}} = 0 \checkmark$$

$\Rightarrow$ $\boxed{\text{Motional Electric Field upstream } \underset{\sim}{E}_1' = 0 \text{ in HTF}}$

6. In general, one can determine $\underset{\sim}{U}_{HT}$ by

$$\boxed{\underset{\sim}{U}_{HT} = \frac{\hat{n} \times (\underset{\sim}{u}_1 \times \underset{\sim}{B}_1)}{\hat{n} \cdot \underset{\sim}{B}_1}}$$

$\leftarrow$ works even when $\underset{\sim}{u}_1$ is not in the NIF. ($\theta_{Un} \neq 0$).

I. E. (Continued)

7. Limit as $\theta_{Bn} \to 90°$

a. $\underset{\sim}{U}_{HT} = U_1 \tan\theta_{Bn} \hat{\underset{\sim}{z}} = U_1 \frac{\sin\theta_{Bn}}{\cos\theta_{Bn}} \hat{\underset{\sim}{z}}$

b. As $\theta_{Bn} \to 90°$, $\underset{\sim}{U}_{HT}$ increases rapidly since $\cos\theta_{Bn} \to 0$.

c. For $\theta_{Bn} > 89°$, a relativistic $\underset{\sim}{U}_{HT}$ is necessary

d. For $\theta_{Bn} = 90°$, it is not possible to define the HTF.

8. Downstream Frame:

a. Since B_n & $\underset{\sim}{E}_t'$ are constant across the shock, it can be shown that $\boxed{\underset{\sim}{U}_2' \parallel \underset{\sim}{B}_2}$ in the HTF.

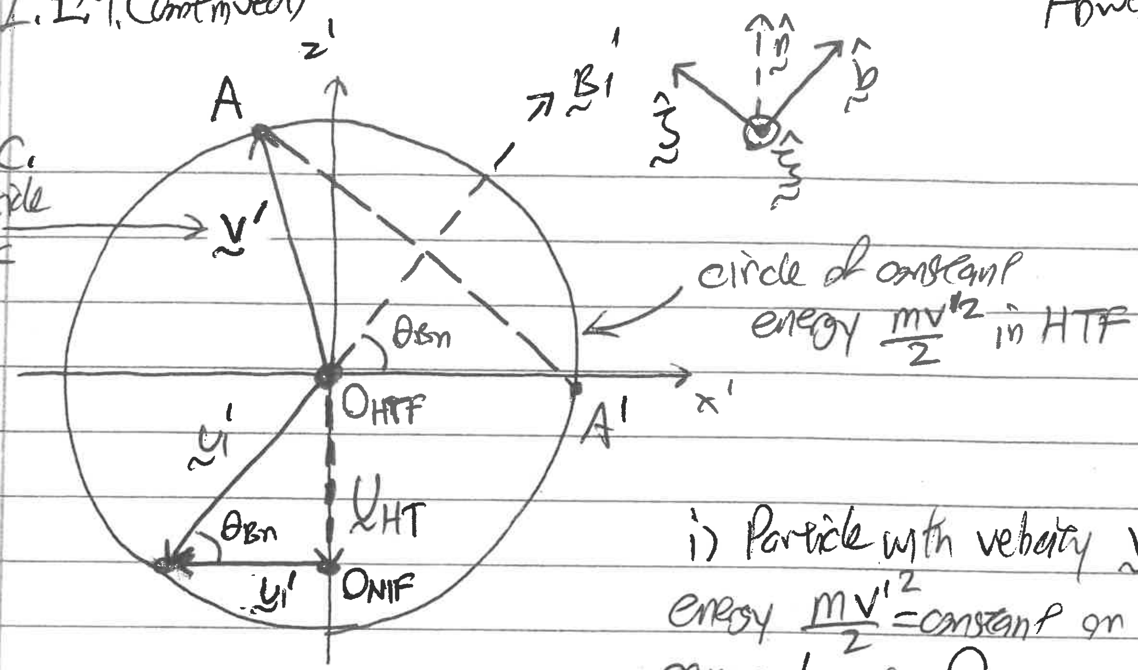
b. Therefore, $\underset{\sim}{E}_2' = \underset{\sim}{U}_2' \times \underset{\sim}{B}_2 = 0$, and there is no motional electric field downstream in HTF, $\boxed{\underset{\sim}{E}_2' = 0}$

9. Particle motion in HTF:

a. Since $\underset{\sim}{E}' = 0$ in the HTF both upstream and downstream, the particle energy is constant, lying on a sphere centered at the origin of the HTF, O_{HTF}.

b. $\underset{\sim}{E}' \neq 0$ only in the shock transition layer.

c. Individual particle velocity in HTF



i) Particle with velocity $\underset{\sim}{v}'$ has energy $\frac{mv'^2}{2}$ = constant on sphere centered at O_{HTF}

ii) Particle gyromotion moves on surface of sphere from A to A' above.

iii) Local magnetic field aligned coordinate system $(\hat{b}, \hat{\zeta}, \hat{\xi})$, where $\hat{n}$ is in the $(\hat{b}, \hat{\zeta})$ plane.

d. Particle dynamics are simplified in HTF, as shown in Schwartz, "Solar wind & the Earth's bow shock," in Priest (ed.) <u>Solar System Magnetic Fields</u>, Dordrecht: Reidel (1985)

and Schwartz, Thomsen, & Gosling, JGR 88: 2039 (1983).

10a. Within the shock layer, out of coplanarity plane components arise in the fields

b. But, the electric field has only $\underset{\sim}{E}' = E_x' \hat{x}$ in the normal direction.

II. Supercriticality

A. First Critical Mach Number

1. At sufficiently low Mach number, the dissipation mechanisms present in MHD can provide sufficient dissipation in the shock layer to slow the inflow, thermalize the released inflow energy, and increase entropy
   a. Resistivity
   b. Viscosity
   c. Heat Conduction

2. But, at sufficiently high Mach number (M_A greater than 2 or 3), these mechanisms are not sufficient to heat the plasma enough so that the downstream sound speed exceeds the downstream flow velocity $\Rightarrow$ subsonic downstream flow!
   ($V_{2n} < V_{f2}$ for fast magnetosonic shock)

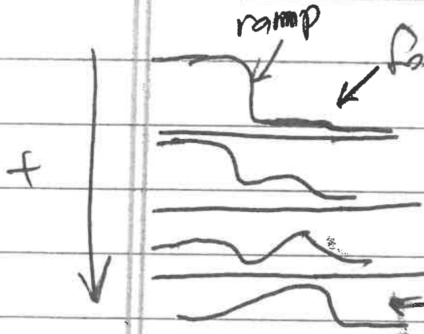
3. For a perpendicular shock ($\theta_{Bn} = 90°$) at $\beta \ll 1$, this critical fast Mach number $M_f \gtrsim M_c = 2.76$.
   Ref: Edmiston & Kennel, J. Plasma Phys, 32:429 (1984)
   a. This $M_c(\beta, \theta_{Bn})$ in MHD Shock theory.
   b. Determined by setting $V_{2n} = V_{f2}$.

4. At $M_f > M_c$, the shock can achieve additional dissipation of incoming flow energy by reflecting a fraction of the inflowing plasma back upstream.

5. The onset of particle reflection at $M_f > M_c$ signifies a breakdown of fluid theory.
   a. Fluid models cannot capture reflection and acceleration of some fraction of incoming ions
   b. A kinetic model must be adopted to describe collisionless plasma shocks.

B. Second Critical Mach Number

1. At yet higher Mach number, the shock structure is observed to become time dependent.
   Ref: Krasnoselskikh et al., Phys. Plasmas 9:1192 (2002).
2. The shock can under periodic reformation.



← new ramp where foot was.

3. An approximate expression is
$$M_f \gtrsim M_{2c} = \left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \cos\theta_{Bn}.$$

C. Heliospheric Shocks

1. Almost all heliospheric shocks (planetary bow shocks, termination shock, interplanetary shocks) are supercritical. Thus, ion reflection plays a key role.

$\Rightarrow$ $\boxed{\text{Supercritical Quasi-perpendicular \& Quasi-parallel shocks}}$