Lecture #2 - Single Particle Motion

I. Overall Framework of Plasma Physics

Particles: Ions & Electrons

position $\underline{x}_s$, velocity $\underline{v}_s$ $\Rightarrow$ ρ_q, $\underline{j}$

Lorentz Force Law

$$m_s \frac{d\underline{v}_s}{dt} = q_s\left(\underline{E} + \underline{v}_s \times \underline{B}\right)$$

Maxwell's Equations

$$\nabla \cdot \underline{E} = \frac{\rho_q}{\epsilon_0}$$

$$\nabla \times \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$\nabla \times \underline{B} = \mu_0 \underline{j} + \mu_0 \epsilon_0 \frac{\partial \underline{E}}{\partial t}$$

$$\nabla \cdot \underline{B} = 0$$

$\underline{B}$ & $\underline{E}$ **Fields**

Single Particle Motion Description

$$\boxed{\text{What we want to study is how charged particles move in prescribed } \underline{E} \text{ \& } \underline{B} \text{ fields.}}$$

II. Larmor Motion: Constant, Uniform $\underline{B}$ with $\underline{E}=0$

A. 1. Nonrelativistic Limit $v \ll c$

2. Drop subscript "s" for species

3. Thus, for $\underline{E}=0$,
$$m\frac{d\underline{v}}{dt} = q\,\underline{v} \times \underline{B}$$

4. Take $\underline{B} = B_0 \hat{\underline{z}}$ were $B_0 = \text{const}$

B. Solution:

1.
$$\frac{dv_x}{dt} = \frac{qB_0}{m} v_y \quad (1)$$

$$\frac{dv_y}{dt} = \frac{-qB_0}{m} v_x \quad (2)$$

$$\frac{dv_z}{dt} = 0 \quad \Rightarrow v_z = \text{constant}$$

2. Define: $\boxed{\text{Cyclotron Frequency: } \Omega \equiv \frac{qB_0}{m}}$

3. To Solve: a. Take $\frac{d}{dt}$ ① and substitute ②

$$\frac{d^2 v_x}{dt^2} = -\Omega^2 v_x$$

b. General Solution: $v_x = A e^{-i\Omega t} + B e^{i\Omega t}$

c. Apply Initial Conditions to Solve for A & B

i. Take $v_x = v_\perp$, $v_y = 0$ at $t=0$. $\Rightarrow A = B = \frac{v_\perp}{2}$

ii. Let $v_z = v_\parallel$ at $t=0$ also.

d. Thus,

$$v_x = v_\perp \cos \Omega t$$
$$v_y = -v_\perp \sin \Omega t$$
$$v_z = v_\parallel$$

e. Solve for position: $\frac{dx}{dt} = \underset{\sim}{v} \Rightarrow$

$$x = \frac{v_\perp}{\Omega} \sin \Omega t + x_0$$
$$y = \frac{v_\perp}{\Omega} \cos \Omega t + y_0$$
$$z = v_\parallel t + z_0$$

4. Define: $\boxed{\text{Larmor Radius } r_L \equiv \frac{v_\perp}{\Omega} = \frac{m v_\perp}{qB_0}}$

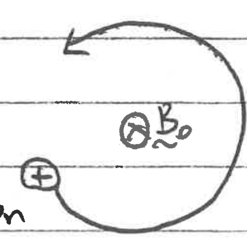
5. Summary:

a. $\underset{\sim}{x}(t) = r_L(\sin \Omega t\, \hat{x} + \cos \Omega t\, \hat{y}) + v_\parallel t \hat{z} + \underset{\sim}{x}_0$

b. $\underset{\sim}{v}(t) = v_\perp(\cos \Omega t\, \hat{x} - \sin \Omega t\, \hat{y}) + v_\parallel \hat{z}$

C. Properties

1. Diamagnetic:



ion

electron

Field due to Larmor motion opposes mean field

II.C. (Continued)

2. Constant Energy:

a. $\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = \underset{\sim}{v} \cdot \left(m\frac{d\underset{\sim}{v}}{dt}\right) = \underset{\sim}{v} \cdot \left[q_s(\underset{\sim}{v} \times \underset{\sim}{B})\right] = 0$

b. Thus, $v_\perp$ = constant.

III. $\underset{\sim}{E} \times \underset{\sim}{B}$ Drift: Constant, Uniform $\underset{\sim}{B}$ and $\underset{\sim}{E}$

A. Drift Motion $\quad \underset{\sim}{B} = B_0 \hat{\underset{\sim}{z}}$

1. $m\frac{d\underset{\sim}{v}}{dt} = q(\underset{\sim}{E} + \underset{\sim}{v} \times \underset{\sim}{B})$

2. What velocity $\underset{\sim}{v}$ leads to RHS $= 0$? $\Rightarrow$ no acceleration $\Rightarrow$ drift

a. $\underset{\sim}{E} = -\underset{\sim}{v} \times \underset{\sim}{B}$

b. Cross with $\underset{\sim}{B}$: $\quad \underset{\sim}{E} \times \underset{\sim}{B} = -(\underset{\sim}{v} \times \underset{\sim}{B}) \times \underset{\sim}{B} = B_0^2(\underset{\sim}{v} - v_z\hat{\underset{\sim}{z}})$

c. Thus,
$$\underbrace{\underset{\sim}{v} - v_z \hat{\underset{\sim}{z}}}_{\text{Perpendicular to } \underset{\sim}{B}_0} = \frac{\underset{\sim}{E} \times \underset{\sim}{B}}{B_0^2}$$

3. Define $\boxed{\text{"E cross B" velocity} \quad \underset{\sim}{v}_E \equiv \frac{\underset{\sim}{E} \times \underset{\sim}{B}}{B_0^2}}$

B. Motion in $\underset{\sim}{E} \times \underset{\sim}{B}$ drift Frame

1. Solve for velocity $\underset{\sim}{u}$ in $\underset{\sim}{E} \times \underset{\sim}{B}$ frame: $\quad \underset{\sim}{v} = \underset{\sim}{u} + \underset{\sim}{v}_E$

2. Substitute for $\underset{\sim}{v}$: $\quad m\frac{d\underset{\sim}{u}}{dt} + \cancel{m\frac{d\underset{\sim}{v}_E}{dt}}^{\,0} = q(\underset{\sim}{E} + \underset{\sim}{v}_E \times \underset{\sim}{B} + \underset{\sim}{u} \times \underset{\sim}{B})$

a. $\underset{\sim}{v}_E \times \underset{\sim}{B} = \frac{(\underset{\sim}{E} \times \hat{\underset{\sim}{z}}) \times \hat{\underset{\sim}{z}}\, \cancel{B_0^2}}{\cancel{B_0^2}} = E_z \hat{\underset{\sim}{z}} - \underset{\sim}{E}$

b. Thus $\quad m\frac{d\underset{\sim}{u}}{dt} = q(E_z \hat{\underset{\sim}{z}} + \underset{\sim}{u} \times \underset{\sim}{B})$

3. Parallel Motion ($\hat{z}$): $\quad m\frac{du_z}{dt} = qE_z \Rightarrow \boxed{u_z = \frac{qE_z}{m}t + u_{z0}}$

III. B. (Continued)

4. <u>Perpendicular Motion:</u> $\underset{\sim}{v}_\perp = \underset{\sim}{v} - v_z \hat{\underset{\sim}{z}}$

a. $m \dfrac{d\underset{\sim}{v}_\perp}{dt} = q(\underset{\sim}{v}_\perp \times \underset{\sim}{B})$ ~~This is identical to the case with~~ $\underset{\sim}{E} = 0$.

b. Thus, $\underset{\sim}{v}_\perp = v_\perp (\cos \Omega t \, \hat{\underset{\sim}{x}} - \sin \Omega t \, \hat{\underset{\sim}{y}})$

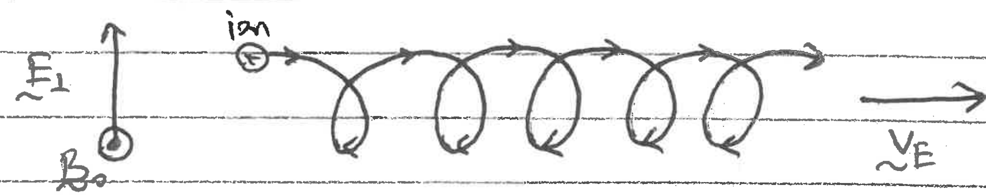
c. In the $\underset{\sim}{E} \times \underset{\sim}{B}$ drift frame, you have the usual Larmor motion

5. Full Solution:

$$\underset{\sim}{v} = \underbrace{\left(\frac{q E_z}{m} t + v_{z0}\right)\hat{\underset{\sim}{z}}}_{\text{Parallel Motion}} + \underbrace{v_\perp(\cos \Omega t\, \hat{\underset{\sim}{x}} - \sin \Omega t\, \hat{\underset{\sim}{y}})}_{\text{Larmor Motion}} + \underbrace{\left(\frac{\underset{\sim}{E} \times \underset{\sim}{B}}{B_0^2}\right)}_{\underset{\sim}{E} \times \underset{\sim}{B} \text{ drift}}$$

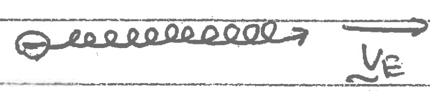
C. <u>Physical Picture:</u>

1.



a. Acceleration by $\underset{\sim}{E}$ $\Rightarrow$ r_L increases
Deceleration by $\underset{\sim}{E}$ $\Rightarrow$ r_L decreases } This asymmetry leads to the drift

2. $\underset{\sim}{E} \times \underset{\sim}{B}$ drift is <u>independent of charge.</u> $\Rightarrow$ No net current due to $\underset{\sim}{E} \times \underset{\sim}{B}$ drift.



IV <u>Multiple Timescale Methods</u>

A. 1. A powerful approach to solving many plasma physics problems is the use of <u>multiple timescale methods</u>.

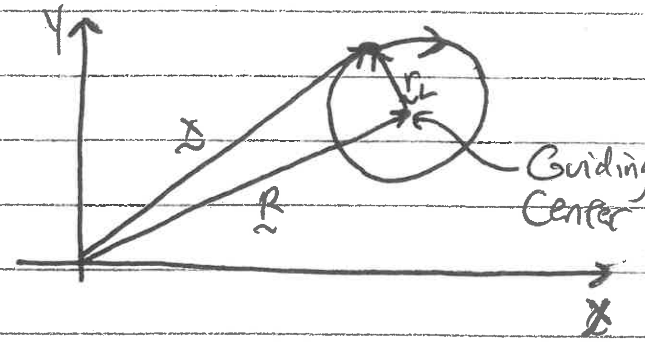
2. In many problems, different components of the motion occur on disparate timescales.

IV. A. 2. (Continued)

a. For Example, $\underset{\sim}{E} \times \underset{\sim}{B}$ drift

Decomposition of Motion:
i. Rapid Larmor motion about field line
ii. Slow drift across field line.

3. Define: Guiding Center



a. Position can be split into Guiding Center $\underset{\sim}{R}$ plus Larmor motion $\underset{\sim}{r}_L$

$$\underset{\sim}{x} = \underset{\sim}{R} + \underset{\sim}{r}_L$$

4. Basic concept for multiscale methods:

a. Average over fast timescale motion:

$$\int_0^{\frac{2\pi}{\Omega}} dt\, \underset{\sim}{r}_L(t) = 0$$

b. This leaves the slow timescale drift motion $\underset{\sim}{R}(t)$.

V. ∇B & Curvature Drifts: Constant, Non-uniform $\underset{\sim}{B}$ fields

A. In the fusion program, magnetic fields used to confine the plasma are neither straight nor uniform. We want to understand particle motion in $\underset{\sim}{B}$ fields of varying strength and curved $\underset{\sim}{B}$ fields.

B. Drift due to a General Force $\underset{\sim}{F}$

1. Analogous to $\underset{\sim}{E} \times \underset{\sim}{B}$ drift, take $m \frac{d\underset{\sim}{v}}{dt} = q \underset{\sim}{v} \times \underset{\sim}{B} + \underset{\sim}{F}$

$$\Rightarrow \boxed{\underset{\sim}{V}_D = \frac{1}{q} \frac{\underset{\sim}{F} \times \underset{\sim}{B}}{B_0^2}}$$

2. NOTE: The direction of this drift depends on the charge sign.

V. (Continued)

C. The ∇B ("GradB") Drift

1. Simplest Case $\quad \nabla|\underset{\sim}{B}| \perp \underset{\sim}{B}$

2. Physical Picture:

a. Stronger $B \Rightarrow$ smaller r_L
   Weaker $B \Rightarrow$ larger r_L

3. <u>Multiscale Approach</u>: a. Small scale: Larmor Radius $r_L = \frac{v_\perp}{\Omega}$
   b. Large scale: $\underset{\sim}{B}$ Scale length $L \equiv \left(\frac{\nabla B}{B}\right)^{-1}$

c. We may use a perturbative approach in the small ~~expansion parameter~~ $\epsilon \equiv \frac{r_L}{L} << 1$

d. We may derive the average force on the particle $\langle F \rangle$ (averaged over the Larmor period $T = \frac{2\pi}{\omega_c}$) due to the ∇B.

$$\langle \underset{\sim}{F} \rangle = -\frac{q v_\perp^2}{2\Omega} \nabla B$$

4. The result is the ∇B drift, $\boxed{\underset{\sim}{V}_{\nabla B} = \frac{-v_\perp^2}{2\Omega} \frac{\nabla B \times \underset{\sim}{B}}{B^2}}$

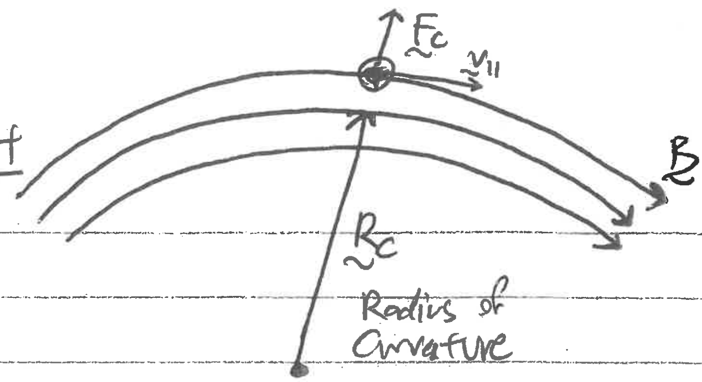
a. <u>NOTE</u>: Since $\Omega = \frac{qB}{m}$, the ∇B drift depends on charge.
   $\Rightarrow$ Ions and electrons drift in opposite directions

b. Drift magnitude depends on perpendicular energy $\frac{1}{2} m v_\perp^2$.

V. (Continued)

D. Curvature Drift

1. Physical Picture



2. Simple Estimate:

a. For a particle moving along a circular path along $\underset{\sim}{B}$, the centrifugal force felt by the particle is

$$\underset{\sim}{F_c} = \frac{m v_\parallel^2}{R_c}\underset{\sim}{\hat{r}} = \frac{m v_\parallel^2}{R_c^2}\underset{\sim}{R_c}$$

b. Treating this as the general force $\underset{\sim}{F}$, we find

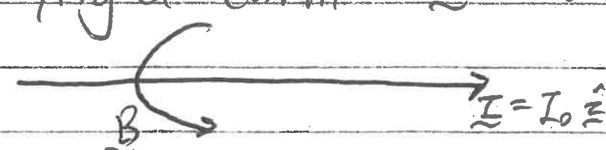
Curvature Drift $$\boxed{\underset{\sim}{V_c} \equiv \frac{m v_\parallel^2}{q B^2}\frac{\underset{\sim}{R_c}\times\underset{\sim}{B}}{R_c^2}} = \frac{v_\parallel^2}{\Omega B}\frac{\underset{\sim}{R_c}\times\underset{\sim}{B}}{R_c^2}$$

3. Properties:
a. Depends on parallel energy $\frac{1}{2}m v_\parallel^2$
b. Again, ions & electrons drift in opposite directions

4. NOTE:
When $\underset{\sim}{B}$ field lines are curved, there is typically also a gradient in $|\underset{\sim}{B}|$, so both ∇B & curvature drifts will be important.

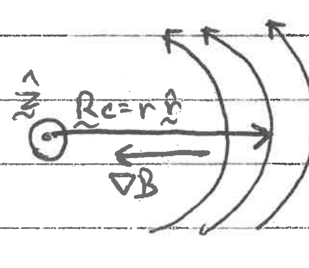
E. Example: Current Carrying Wire

Consider a wire carrying a current $\underset{\sim}{I} = I_0\underset{\sim}{\hat{z}}$



1. In cylindrical coordinates (r,ϕ,z), $\underset{\sim}{B} = \frac{\mu_0 I_0}{2\pi r}\underset{\sim}{\hat{\phi}}$

2. End on view



From NRL Plasma Formulary p.6,

$$\nabla B = \frac{\partial B}{\partial r}\underset{\sim}{\hat{r}} + \cancel{\frac{1}{r}\frac{\partial B}{\partial\phi}\underset{\sim}{\hat{\phi}}} + \cancel{\frac{\partial B}{\partial z}\underset{\sim}{\hat{z}}} = -\frac{\mu_0 I_0}{2\pi r^2}\underset{\sim}{\hat{r}}$$

3. ∇B Drift:

$$\underset{\sim}{V}_{\nabla B} = -\frac{v_\perp^2}{2\Omega}\frac{\nabla B\times \underset{\sim}{B}}{B_0^2} = -\frac{v_\perp^2}{2\Omega}\frac{\left(-\frac{\mu_0 I_0}{2\pi r^2}\hat{\underset{\sim}{r}}\right)\times\left(\frac{\mu_0 I_0}{2\pi r}\hat{\underset{\sim}{\phi}}\right)}{\left(\frac{\mu_0 I_0}{2\pi r}\right)^2} = +\frac{v_\perp^2}{2\Omega r}\hat{\underset{\sim}{z}}$$

4. Curvature Drift:

$$\underset{\sim}{V}_c = \frac{v_\parallel^2}{\Omega B}\frac{\underset{\sim}{R}_c\times \underset{\sim}{B}}{R_c^2} = \frac{v_\parallel^2}{\Omega\left(\frac{\mu_0 I_0}{2\pi r}\right)}\frac{(r\hat{\underset{\sim}{r}})\times\left(\frac{\mu_0 I_0}{2\pi r}\hat{\underset{\sim}{\phi}}\right)}{r^2} = \frac{v_\parallel^2}{\Omega r}\hat{\underset{\sim}{z}}$$

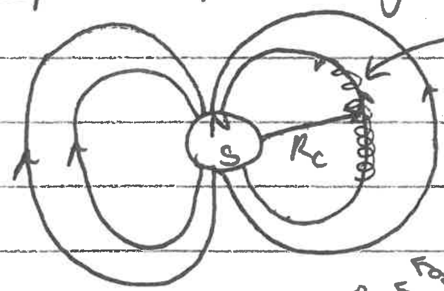
5. Net Drift: $\underset{\sim}{V} = \underset{\sim}{V}_{\nabla B} + \underset{\sim}{V}_c = \frac{1}{\Omega r}\left(\frac{v_\perp^2}{2} + v_\parallel^2\right)\hat{\underset{\sim}{z}}$

a. NOTE: $\frac{1}{\Omega r} = \frac{m}{qBr} = \frac{m\, 2\pi \cancel{r}}{q \mu_0 I_0 \cancel{r}}$, so $\boxed{\underset{\sim}{V} = \frac{2\pi}{q\mu_0 I_0}\left(\frac{m v_\perp^2}{2} + m v_\parallel^2\right)\hat{\underset{\sim}{z}}}$

Velocity is independent of r!

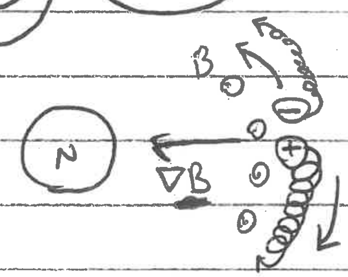
F. Example: Earth's Magnetosphere

Side View:



1. Particles trapped in Earth's dipole field experience ∇B & curvature drifts

Top View:



2. These drifts produce the "ring current" in the westward direction

3. Strength of ring current is proportional to the energy of the particles.

$\Rightarrow$ Magnetic Storms!