Lecture #21: Magnetic Reconnection in Resistive MHD

I. Resistive Magnetohydrodynamics: Equations & Properties

A. Equations of Resistive MHD

1. Continuity: $\frac{\partial \rho}{\partial t} + \underset{\sim}{u} \cdot \nabla \rho = -\rho \nabla \cdot \underset{\sim}{u}$

2. Momentum: $\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} = -\frac{1}{\rho} \nabla \left(p + \frac{B^2}{2\mu_0}\right) + \frac{\underset{\sim}{B} \cdot \nabla \underset{\sim}{B}}{\mu_0 \rho} + \nu \nabla^2 \underset{\sim}{u}$

3. Magnetic Induction: $\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{u} \times \underset{\sim}{B}) + \frac{\eta}{\mu_0} \nabla^2 \underset{\sim}{B}$

4. Adiabatic Eq. of State: $\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0$

where Def: Kinematic viscosity $\nu \equiv \frac{\mu}{\rho}$ ← coefficient of shear viscosity

Def: Resistivity $\eta \equiv \frac{m_e \nu_{ei}}{e^2 n_0}$ ← electron-ion collision rate

5. Additional Equations used to derive Resistive MHD

a. Faraday's Law $\frac{\partial \underset{\sim}{B}}{\partial t} = -\nabla \times \underset{\sim}{E}$

b. Ampere's Law $\underset{\sim}{j} = \frac{1}{\mu_0} \nabla \times \underset{\sim}{B}$

c. No Magnetic Divergence $\nabla \cdot \underset{\sim}{B} = 0$

d. Coulomb's Law $\nabla \cdot \underset{\sim}{E} = 0$ ← zero charge density ρ_q

I. A. (Continued)

6. Ohm's Law:

a. In general, Ohm's Law can be derived from the two-fluid momentum equation for electrons where current density $\underset{\sim}{j} \equiv en(\underset{\sim}{u}_i - \underset{\sim}{u}_e)$

b. Generalized Ohm's Law

$$\underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B} = \underbrace{\eta \underset{\sim}{j}}_{\text{Resistivity}} + \underbrace{\frac{1}{en} \underset{\sim}{j} \times \underset{\sim}{B}}_{\text{Hall term}} - \underbrace{\frac{1}{en} \nabla \cdot \underset{\approx}{P}_e}_{\text{Electron Pressure tensor}} + \underbrace{\frac{m_e}{e^2 n} \left[\frac{\partial \underset{\sim}{j}}{\partial t} + \nabla \cdot (\underset{\sim}{j}\underset{\sim}{u} + \underset{\sim}{u}\underset{\sim}{j}) \right]}_{\text{Electron inertia}}$$

c. Resistive Ohm's Law

$$\boxed{\underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B} = \eta \underset{\sim}{j}}$$

B. Frozen-In Magnetic Field:

1. In the absence of resistivity, the Ideal Ohm's Law

$$\boxed{\underset{\sim}{E} + \underset{\sim}{u} \times \underset{\sim}{B} = 0}$$

implies that the magnetic field is frozen-in to the plasma flow

$\Rightarrow$ • The plasma cannot slip relative to the magnetic field. Magnetic flux is conserved!

2. Note that this implies $E_{\parallel} = 0$.

3a. A necessary condition for magnetic reconnection is some non-ideal term to break the magnetic field.

I. B. (Continued)

3. (Continued)

b. In resistive MHD, resistivity breaks flux conservation, allowing annihilation of magnetic flux and $E_{\parallel} = \eta j_{\parallel} \neq 0$.

C. Resistive Diffusion vs. Magnetic Reconnection

1. In the presence of $\eta \neq 0$, magnetic flux can be annihilated by either resistive diffusion or through magnetic reconnection.

a. Must have an appropriate geometry for magnetic reconnection to occur.

b. Timescales of both process are distinct.

2. Timescales:

a) Def: Alfven Time, τ_A

$$\boxed{\tau_A = \frac{L}{v_A}}$$

← macroscopic scale length of change in $\underset{\sim}{B}$

b) Def: Resistive Diffusion Time, τ_R

i) $\frac{\partial \underset{\sim}{B}}{\partial t} \sim \frac{\eta}{\mu_0} \nabla^2 \underset{\sim}{B} \Rightarrow \frac{B}{\tau_R} \sim \frac{\eta B}{\mu_0 L^2}$

ii) $$\boxed{\tau_R = \frac{\mu_0 L^2}{\eta}}$$

I. C Continued

3. Def: Lundquist Number, S

a. $S \equiv \frac{\tau_R}{\tau_A} = \frac{\mu_0 L^2/\eta}{L/v_A} = \frac{\mu_0 v_A L}{\eta}$ $\quad \boxed{S = \frac{\mu_0 v_A L}{\eta}}$

b. Ratio of resistive diffusion to Alfvenic timescales

c. Similar to magnetic Reynolds number $Re_M = \frac{\mu_0 U L}{\eta}$, but with v_A replacing U

d. For fusion or astrophysical plasma $S \gtrsim 10^8$
$\Rightarrow$ resistive diffusion is very slow.

4. Def: Magnetic Reconnection Timescale, τ_r

a. We will find $\tau_A \ll \tau_r \ll \tau_R$

b. Magnetic reconnection enables magnetic flux annihilation much faster than resistive diffusion.

c. Magnetic reconnection is only possible when the geometry of field lines allows it — change of magnetic field direction.

D. Forced vs. Spontaneous Magnetic Reconnection

1. If magnetic field changes direction, reconnection can occur!

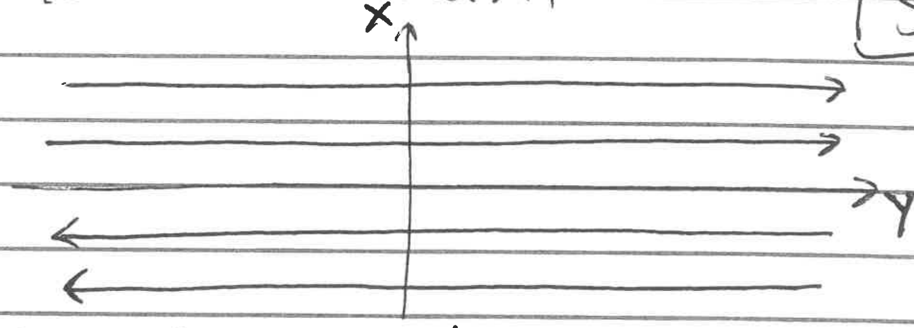
a. Forced magnetic reconnection: When external driving pushes oppositely directed magnetic fields together $\Rightarrow$ Sweet-Parker Model

b. Spontaneous magnetic reconnection: Driven by Tearing Instability.

II. Sweet-Parker Model of Magnetic Reconnection

A. Equilibrium Magnetic Field Configuration

1. $\underset{\sim}{B}_0 = B_{1_0} \tanh\left(\frac{x}{a}\right) \hat{\underset{\sim}{y}} + B_{z_0} \hat{\underset{\sim}{z}}$ $\boxed{\text{Harris Current Sheet}}$

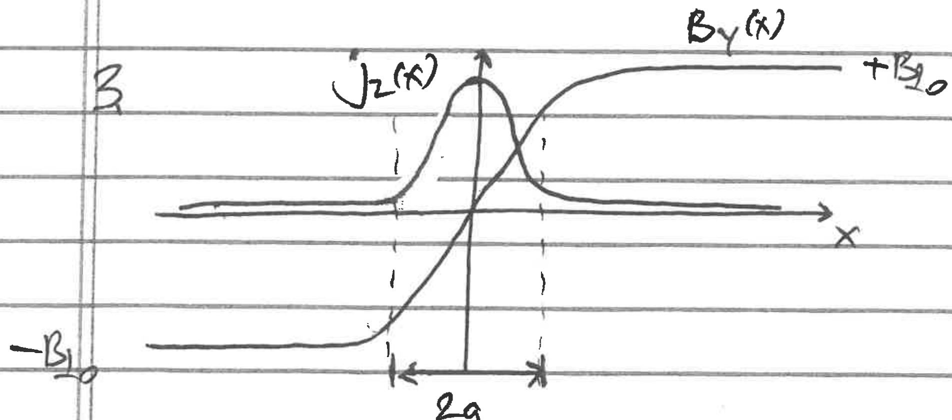


2. Ampere's Law: $\underset{\sim}{j} = \frac{1}{\mu_0} \nabla \times \underset{\sim}{B}$

a. $\frac{1}{\mu_0} \nabla \times \underset{\sim}{B} = \frac{1}{\mu_0} \begin{vmatrix} \hat{\underset{\sim}{x}} & \hat{\underset{\sim}{y}} & \hat{\underset{\sim}{z}} \\ \partial_x & \cancel{\partial_y}^{\,0} & \cancel{\partial_z}^{\,0} \\ \cancel{B_x}^{\,0} & B_y & B_z \end{vmatrix} = \frac{1}{\mu_0} \frac{\partial B_y}{\partial x} \hat{\underset{\sim}{z}} - \cancel{\frac{\partial B_z}{\partial x}}^{\,0} \hat{\underset{\sim}{y}}$

b. $j_z = \frac{1}{\mu_0} \frac{\partial B_y}{\partial x} = \frac{B_{1_0}}{\mu_0 a} \operatorname{sech}^2\left(\frac{x}{a}\right)$ Using $\frac{d}{dx} \tanh u = \operatorname{sech}^2 u \frac{du}{dx}$

3.



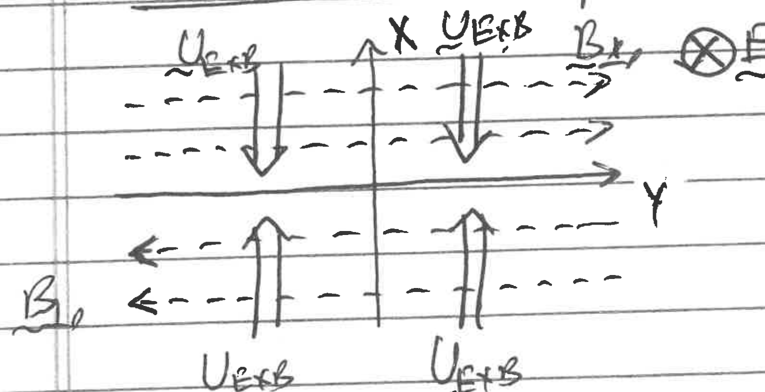
4. Pressure Balance: Note that in equilibrium, the Harris Sheet requires a pressure variation $p_0(x)$ to balance magnetic pressure along x, $\frac{\partial}{\partial x}\left(p_0(x) + \frac{|\underset{\sim}{B}_0|^2}{2\mu_0}\right) = 0.$

B. Sweet-Parker Model

Refs: Sweet, Nuovo Cimento Suppl. Ser X, 8: 188 (1958)
Parker, J. Geophys. Res., 62: 509 (1957)

1. This has been a foundational model for resistive magnetic reconnection for six decades, with validation by computer simulations and laboratory experiments.

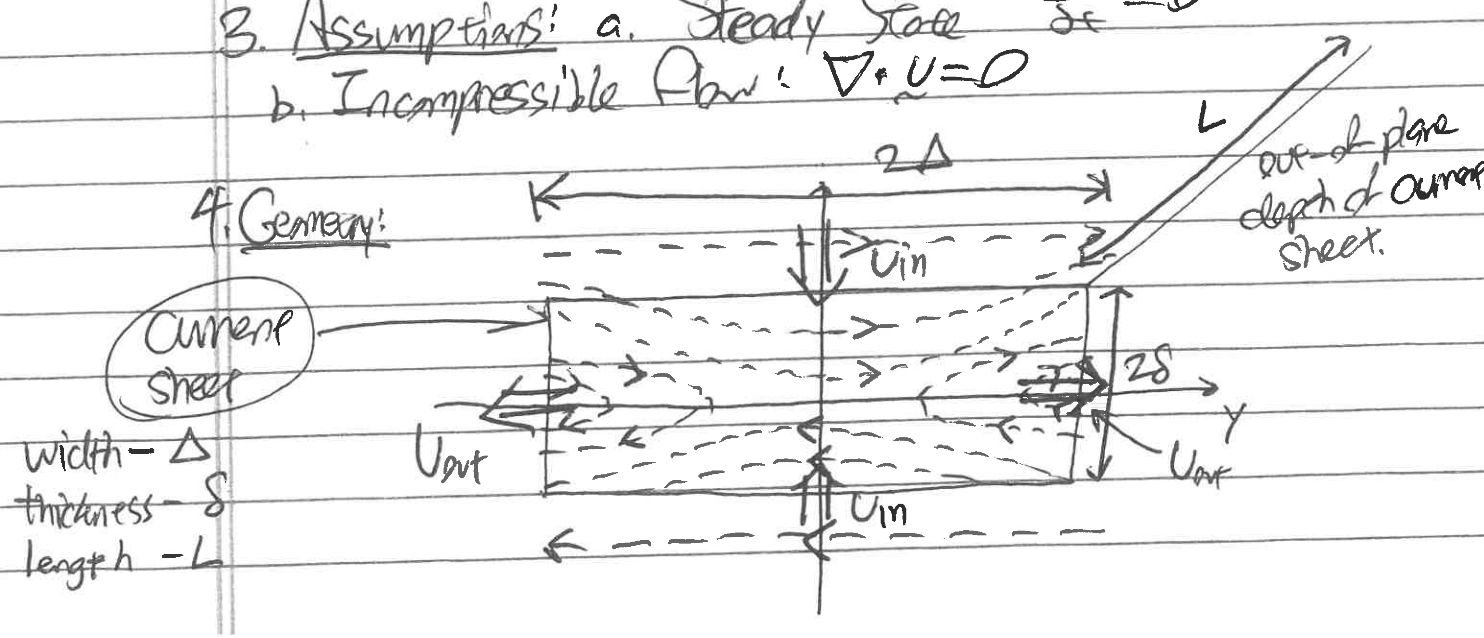
2. Forced Reconnection: a) Impose $\mathbf{E} = E_z \hat{z} > 0$,



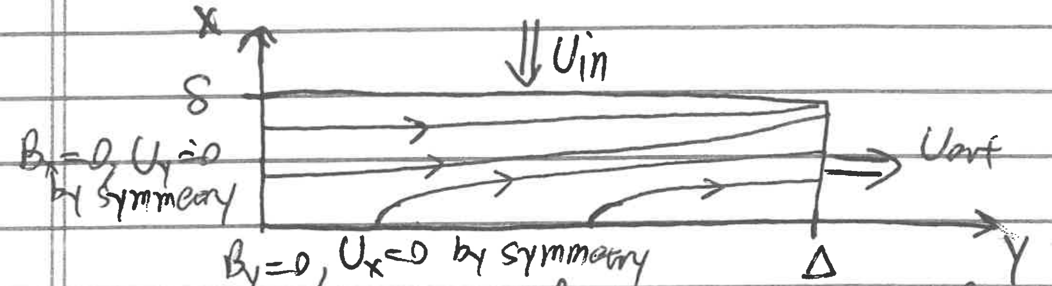
b. What is the inflow speed U_{in} at which oppositely directed (in y) magnetic fields merge and annihilate?

3. Assumptions: a. Steady State $\frac{\partial}{\partial t} = 0$
b. Incompressible flow: $\nabla \cdot \mathbf{U} = 0$

4. Geometry:



5. Consider one quadrant of the current sheet region:



a. Incompressible: $\cancel{\frac{\partial \rho}{\partial t}}^{\,0} + \underset{\sim}{U}\cdot\nabla\rho = -\rho\cancel{\nabla\cdot\underset{\sim}{U}}^{\,0}$

$$\Rightarrow \underset{\sim}{U}\cdot\nabla\rho = 0 \quad\Rightarrow\quad \rho = \text{constant} = \rho_0$$

b. Mass flux into and out of quadrant:

i) $\cancel{\rho_0} U_{in}\Delta \cancel{L} = \cancel{\rho_0} U_{out}\delta \cancel{L}$

ii) $\boxed{U_{in} = U_{out}\frac{\delta}{\Delta}}$ (I)

6. Momentum Equation along $x=0$ from $y=0$ to $y=\Delta$:

a. By symmetry: $U_x=0$, $B_y=0$, $\frac{\partial}{\partial x}=0$

y-component: b. $\cancel{\frac{\partial \underset{\sim}{U}}{\partial t}}^{\,0} + \underset{\sim}{U}\cdot\nabla\underset{\sim}{U} = -\frac{1}{\rho_0}\nabla\left(p+\frac{B^2}{2\mu_0}\right) + \frac{\underset{\sim}{B}\cdot\nabla\underset{\sim}{B}}{\mu_0\rho_0} + \cancel{\nu}\nabla^2\underset{\sim}{U}$ Neglect viscosity

$$U_y\frac{\partial U_y}{\partial y} = -\frac{1}{\rho_0}\frac{\partial}{\partial y}\left(p+\frac{B^2}{2\mu_0}\right) + \cancel{\frac{\left(B_x\frac{\partial}{\partial x}\right)^0\underset{\sim}{B}}{\mu_0\rho_0}}$$

c. $\frac{\partial}{\partial y}\left[\rho_0\frac{U_y^2}{2} + p + \frac{B^2}{2\mu_0}\right] = 0$

d. Integrate from 0 to Δ in y: $\int_0^\Delta dy\frac{\partial}{\partial y}\left[\rho_0\frac{U_y^2}{2}+p+\frac{B^2}{2\mu_0}\right] = 0$

II. B.6. (Continued) Howes 8

e. $\rho_0 \frac{U_y^2(\Delta)}{2} + p(\Delta) + \frac{B_y^2(\Delta) + \cancel{B_{z0}^2(\Delta)}}{2\mu_0} - \cancel{\rho\frac{U_y^2(0)}{2}} - p(0) - \frac{B_y^2(0) + \cancel{B_{z0}^2(0)}}{2\mu_0} = 0$

i) B_{z0} = const, so cancels.

ii) $U_y(0) = 0$

iii) $B_y(0) = 0$ $\quad B_y(\Delta) = B_{\perp 1}$ ← downstream magnitude

iv) $p(\Delta) = p_0$ ← where pressure gradient ends

v) $U_y(\Delta) = U_{out}$ $\quad$ vi) $p(0) = p_{max}$

f. $\rho_0 \frac{U_{out}^2}{2} + p_0 + \frac{B_{\perp 1}^2}{2\mu_0} - p_{max} = 0$

g. $\boxed{\rho_0 \frac{U_{out}^2}{2} = (p_{max} - p_0) - \frac{B_{\perp 1}^2}{2\mu_0}}$ II

7. Momentum Equation along $y=0$ from $x=0$ to $x=\delta$:

~~x-component~~; a. By Symmetry: $U_y = 0$, $B_x = 0$, $\frac{\partial}{\partial y} = 0$

b. $\cancel{\frac{\partial U_x}{\partial t}} + U_x \frac{\partial U_x}{\partial x} = -\frac{1}{\rho_0}\frac{\partial}{\partial x}\left(p + \frac{B^2}{2\mu_0}\right) + \cancel{\frac{(B_y \frac{\partial}{\partial y}) B_x}{\mu_0 \rho_0}} + \cancel{\nu \nabla^2 U_x}$

c. $\frac{\partial}{\partial x}\left[\rho_0 \frac{U_x^2}{2} + p + \frac{B^2}{2\mu_0}\right] = 0$

d. Integrate from $x=0$ to $x=\delta$ $\quad \int_0^\delta dx \frac{\partial}{\partial x}[\ldots] = 0$

e. $\rho_0 \frac{U_x^2(\delta)}{2} + p(\delta) + \frac{B_y^2(\delta)}{2\mu_0} - \cancel{\rho_0 \frac{U_x^2(0)}{2}} - p(0) - \cancel{\frac{B_y^2(0)}{2\mu_0}} = 0$

i) $U_x(\delta) = U_{in}$ $\quad U_x(0) = 0$

ii) $B_y(\delta) = B_{10}$ $\quad B_y(0) = 0$

iii) $p(\delta) = p_0$ $\quad p(0) = p_{max}$

II. B.7. (Continued)

f. $$\boxed{\rho_0 \frac{U_{in}^2}{2} = p_{max} - p_0 - \frac{B_{1o}^2}{2\mu_0}} \quad \text{(II)}$$

8. Ansatz 1: $\rho_0 \frac{U_{in}^2}{2} \ll \frac{B_{1o}^2}{2\mu_0}$ ← Verify a posteriori.

a. Thus $p_{max} - p_0 = \frac{B_{1o}^2}{2\mu_0}$

b. Plug into (II) for $p_{max} - p_0$:

$$\rho_0 \frac{U_{out}^2}{2} = \frac{B_{1o}^2}{2\mu_0} - \frac{B_{1i}^2}{2\mu_0}$$

c. Ansatz 2: $B_{1i}^2 \ll B_{1o}^2$ ← verify a posteriori

d. Thus $$\boxed{\rho_0 \frac{U_{out}^2}{2} = \frac{B_{1o}^2}{2\mu_0}}$$

e. We can show this gives $\boxed{U_{out} = \frac{B_{1o}}{\sqrt{\mu_0 \rho_0}} = V_{A\perp}}$

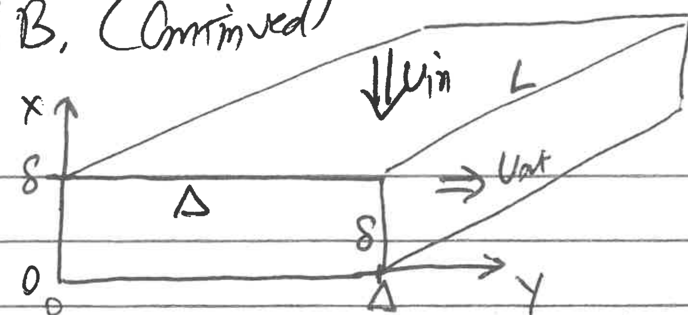
Outflow is a perpendicular Alfven velocity

9. Revisiting (I), using $U_{out} = V_{A\perp}$, $\boxed{U_{in} = V_{A\perp} \frac{\delta}{\Delta}}$

a. We just need to determine the aspect ratio δ/Δ!

II. B. (Continued)

10.



a) In the quadrant, we demand Ohmic dissipation must balance difference in energy outflow from inflow.

b) In the quadrant volume $\Delta \delta L$,

$$u_{in}\frac{B_{\perp 0}^2}{2\mu_0}\Delta \cancel{L} - u_{out}\frac{B_{\perp 1}^2}{2\mu_0}\delta\cancel{L} = \eta j^2 \Delta\delta \cancel{L}$$

c) Ohmic Dissipation: i) $\underline{E} + \underline{u}\times\underline{B} = \eta \underline{j}$

ii) Dot with $\underline{j}$: $\underbrace{\underline{E}\cdot\underline{j}}_{\text{rate of work done/vol}} = -(\underline{u}\times\underline{B})\cdot\underline{j} + \underbrace{\eta|\underline{j}|^2}_{\text{resistive dissipation/vol}}$

iii) ηj_z^2 dominates resistive dissipation

d) Using $u_{in}\Delta = u_{out}\delta$, Again, take $B_{\perp 1}^2 \ll B_{\perp 0}^2$

i) $u_{in}\cancel{\Delta}\left[\frac{B_{\perp 0}^2}{2\mu_0} - \cancelto{0}{\frac{B_{\perp 1}^2}{2\mu_0}}\right] = \eta j^2 \cancel{\Delta}\delta$

ii) $\boxed{u_{in}\frac{B_{\perp 0}^2}{2\mu_0} = \eta j^2 \delta}$

e) From Ampere's Law (See Sec. II A, 2.b.), $j_z = \frac{1}{\mu_0}\frac{\partial B_y}{\partial x} = \frac{1}{\mu_0}\frac{\Delta B_y}{\Delta x}$

$$\boxed{j_z = \frac{B_{\perp 0}}{\mu_0 \delta}}$$

f) $U_{in} \frac{\cancel{B_{10}^2}}{2\cancel{\mu_0}} = \eta \frac{\cancel{B_{10}^2}}{\mu_0^{\cancel{2}} \delta^{\cancel{2}}} \delta \Rightarrow \boxed{U_{in} = \frac{2\eta}{\mu_0 \delta}}$

11. Combining $U_{in} = \frac{2\eta}{\mu_0 \delta}$ and $U_{in} = V_{A1} \frac{\delta}{\Delta}$,

a. $V_{A1} \frac{\delta}{\Delta} = \frac{2\eta}{\mu_0 \delta} \Rightarrow \delta^2 = \left(\frac{2\eta}{\mu_0 V_{A1}}\right) \Delta$

b. $\delta = \frac{\sqrt{2}\,\Delta^{\frac{1}{2}}\Delta^{\frac{1}{2}}}{\left(\frac{\mu_0 V_{A1} \Delta}{\eta}\right)^{\frac{1}{2}}} \Rightarrow \boxed{\frac{\delta}{\Delta} = \frac{\sqrt{2}}{S^{\frac{1}{2}}}}$

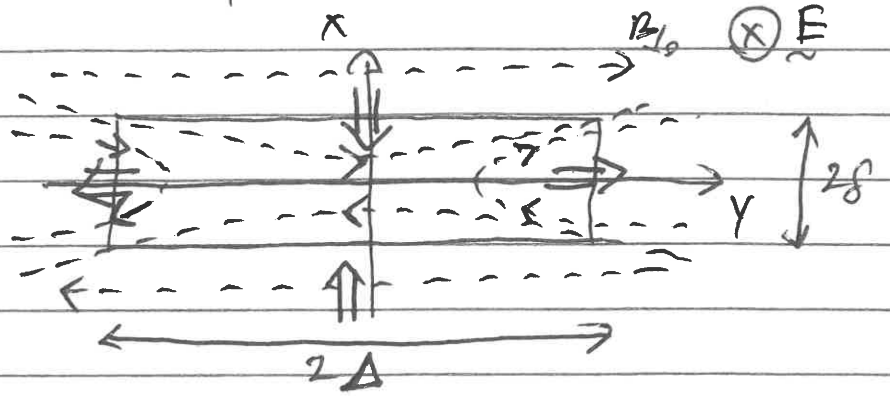
Aspect ratio of current sheet must have $\frac{\delta}{\Delta} \ll 1$ to balance energy inflow with energy dissipation for $S \gg 1$ plasmas

12. Find inflow velocity U_{in}:

a. $U_{in} = V_{A1} \frac{\delta}{\Delta} \Rightarrow \boxed{U_{in} = \frac{\sqrt{2}\, V_{A1}}{S^{\frac{1}{2}}}} \ll V_{A1} = U_{out}$

C. Summary of Sweet-Parker Model

1\.



2\. a. $U_{out} = V_{A\perp} = \frac{B_{\perp 0}^2}{\mu_0 \rho_0}$

b. $U_{in} = \frac{\sqrt{2}}{S^{+\frac{1}{2}}} V_{A\perp}$ where $S \equiv \frac{\mu_0 V_{A\perp} \Delta}{\eta}$ Lundquist Number

c. $\frac{\delta}{\Delta} = \frac{\sqrt{2}}{S^{\frac{1}{2}}}$

3\. Check Ansatz 1: $\rho_0 \frac{U_{in}^2}{2} \ll \frac{B_{\perp 0}^2}{2\mu_0}$

a. $\rho_0 \frac{\cancel{2} V_{A\perp}^2}{\cancel{2} S} = \cancel{\rho_0} \frac{B_{\perp 0}^2}{\mu_0 \cancel{\rho_0}} \frac{1}{S} = \underbrace{\frac{2}{S} \frac{B_{\perp 0}^2}{2\mu_0} \ll \frac{B_{\perp 0}^2}{2\mu_0}}$

b. This holds if $\boxed{S \gg 1}$ ✓

4\. Check Ansatz 2: $B_{\perp 1}^2 \ll B_{\perp 0}^2$

a. $\rho_0 \frac{U_{out}^2}{2} = \frac{B_{\perp 0}^2}{2\mu_0} - \frac{B_{\perp 1}^2}{2\mu_0} \Rightarrow \mu_0 \rho_0 U_{out}^2 = B_{\perp 0}^2 - B_{\perp 1}^2$

b. $B_{\perp 1}^2 = B_{\perp 0}^2 - \mu_0 \rho_0 U_{out}^2 = B_{\perp 0}^2 - \cancel{\mu_0 \rho_0} \left[\frac{B_{\perp 0}^2}{\cancel{\mu_0 \rho_0}} \right] = 0$ ✓

II. (Continued)

D. Revisit Timescales

1. For the current sheet of width Δ,

$$\boxed{\tau_A = \frac{\Delta}{V_{A\perp}}} \qquad \boxed{\tau_R = \frac{\mu_0 \Delta^2}{\eta}}$$

2. Reconnection timescale $\tau_r = \dfrac{\Delta}{U_{in}} = \boxed{\dfrac{S^{\frac{1}{2}}\Delta}{\sqrt{2}\, V_{A\perp}} = \tau_r}$

3. Relative to τ_A: $\boxed{\dfrac{\tau_A}{\tau_A} \sim 1, \quad \dfrac{\tau_r}{\tau_A} \sim S^{\frac{1}{2}}, \quad \dfrac{\tau_R}{\tau_A} \sim S}$

   a. Thus $\boxed{\tau_A \ll \tau_r \ll \tau_R}$ Reconnection occurs on an intermediate timescale

4. Hybrid timescale:

   a. $$(\tau_A \tau_R)^{\frac{1}{2}} = \left[\tau_A^2 \frac{\tau_R}{\tau_A}\right]^{\frac{1}{2}} = \tau_A S^{\frac{1}{2}} = \sqrt{2}\left(\frac{S^{\frac{1}{2}}\Delta}{\sqrt{2}\, V_{A\perp}}\right) = \sqrt{2}\, \tau_r$$

   b. Thus $\boxed{\tau_r \propto (\tau_A \tau_R)^{\frac{1}{2}}}$

5. For laboratory & astrophysical plasmas, $10^4 \lesssim S \lesssim 10^{12}$

   a. Ex: Solar flares in solar corona: $S \sim 10^{10}$

      i) Observed flare magnetic energy release time $\sim$ 20 min.

      ii) Predicted Sweet-Parker $\tau_r \sim$ 2-3 years.

   b. $\Rightarrow$ Some other mechanism enables fast reconnection!