Lecture #22: Resistive Tearing Instability

I. Linear Resistive Tearing Instability

Refs: 1) Richard Fitzpatrick, Plasma Physics: An Introduction, CRC Press; (2014) Sec 5.15 Linear Tearing Mode Theory
2) Furth, Killeen, & Rosenbluth, Phys Fluids, 6: 459 (1963)

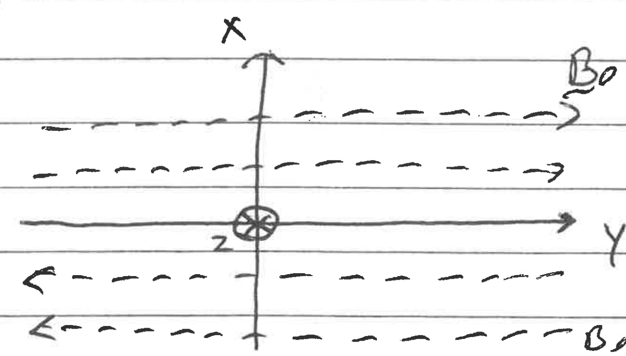
A. Spontaneous Magnetic Reconnection

1. Last time we explored forced reconnection with the Sweet-Parker model.

2. For a general equilibrium magnetic field $\underset{\sim}{B}_0 = B_{0y}(x)\hat{y}$, where $B_{0y}(-x) = -B_{0y}(x)$ (Changing direction through $x=0$), one can determine an equilibrium state with $p_0(x)$ to achieve total pressure balance along x:

$$\frac{\partial}{\partial x}\left(p_0(x) + \frac{B_{0y}^2(x)}{2\mu_0}\right) = 0$$

a. For $\eta = 0$, this equilibrium is stable.



b. But, when $\eta \neq 0$, an instability will develop, relaxing the configuration to lower magnetic energy. ⇒ $\boxed{\text{Tearing Instability}}$

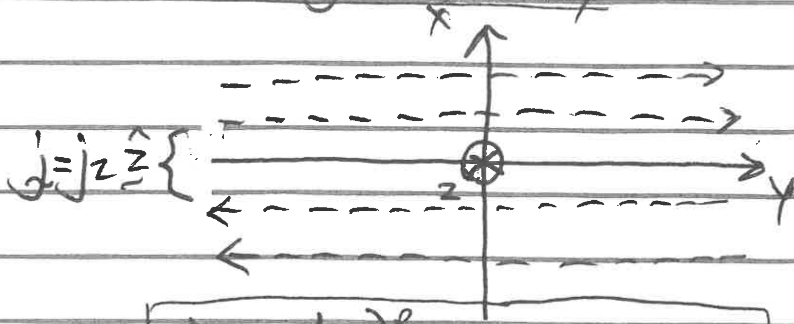
c. Magnetic reconnection occurs in the process of relaxation

3. Released magnetic energy ultimately heats the plasma

I. (Continued) Howes ②

B. Setup for Linear Resistive Tearing Instability

1. a) $\boxed{\underset{\sim}{B}_0 = B_{y0}(x)\,\hat{\underset{\sim}{y}}}$



b. $B_{0y}(-x) = -B_{0y}(x)$

c. $B_0 = \left| \lim_{x \to \pm\infty} B_{0y}(x) \right|$

2. Current: By Ampere's Law : $\boxed{\underset{\sim}{j} = \frac{1}{\mu_0} \frac{\partial B_y}{\partial x} \hat{\underset{\sim}{z}} = j_z \hat{\underset{\sim}{z}} > 0}$

3. Assume: a. $\underset{\sim}{U}_0 = 0$ (No equilibrium flow)

b. $\nabla \cdot \underset{\sim}{U} = 0$ Incompressible Motions

4. Linearized equations of resistive MHD

a. (A1) $\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{U} \times \underset{\sim}{B}_0) + \frac{\eta}{\mu_0} \nabla^2 \underset{\sim}{B}$

b. (B1) $\rho_0 \frac{\partial \underset{\sim}{U}}{\partial t} = -\nabla p + \frac{(\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B}_0}{\mu_0} + \frac{(\nabla \times \underset{\sim}{B}_0) \times \underset{\sim}{B}}{\mu_0}$

c. $\nabla \cdot \underset{\sim}{B} = 0$

d. $\nabla \cdot \underset{\sim}{U} = 0$

Here $\underset{\sim}{B}$ and $\underset{\sim}{U}$ are the perturbed magnetic field & fluid velocity

C. Instability Analysis

1. Assume all perturbed quantities ($\underset{\sim}{B}$ & $\underset{\sim}{U}$) vary as

$$A(x, y, z, t) = A(x)\, e^{iky + \gamma t}$$

a. Converts PDEs to ODEs

b. Does not include $B_{0y}(x)$.

2. a. $\gamma \equiv$ instability growth rate
b. k is wavenumber along $\hat{y}$ direction (periodic in y)
c. $\frac{\partial}{\partial z} = 0$ (2D dynamics)

3. x-component of (A1) and z component of $\nabla \times$ (B1) yield (curl eliminates ∇p)

a. $$\boxed{\gamma B_x = i k B_{0y} U_x + \frac{\eta}{\mu_0}\left(\frac{d^2}{dx^2} - k^2\right) B_x} \quad (A2)$$

b. $$\boxed{\gamma \rho_0 \left(\frac{d^2}{dx^2} - k^2\right) U_x = \frac{i k B_{0y}}{\mu_0}\left(\frac{d^2}{dx^2} - k^2 - \frac{B_{0y}''}{B_{0y}}\right) B_x} \quad (B2)$$

c. NOTE: $B_{0y}'' \equiv \frac{d^2 B_{0y}}{dx^2}$ → "prime indicates $\frac{d}{dx}$"
second derivative of equilibrium $B_{0y}(x)$

4. a. Variables: $B_x(x)$, $U_x(x)$
b. Parameters: $\frac{B_{0y}''}{B_{0y}}$, η, k
c. Eigenvalue: γ

D. Dimensionless Normalization

1. Assume length scale in x is given by a.

e.g. $\underset{\sim}{B_0} = B_0 \tanh\left(\frac{x}{a}\right)\hat{y}$

2. Timescales: a. Alfven time: $\boxed{\tau_A = \frac{a}{v_A}}$ $\quad v_A = \frac{B_{0y}^2}{\sqrt{\mu_0 \rho_0}}$

b. Resistive Diffusion time: $\boxed{\tau_R = \frac{\mu_0 a^2}{\eta}}$

3. Lundquist Number, $\boxed{S \equiv \frac{\tau_R}{\tau_A} = \frac{\mu_0 v_A a}{\eta}}$

I. D. (Continued)

4. Dimensionless Normalization

a. Scalar functions: i) $\psi(x) = \frac{B_x(x)}{B_0}$

ii) $\phi(x) = \frac{i k U_x(x)}{\gamma}$

where $B_0 = \lim_{x\to\infty} B_{0y}(x)$ (upstream magnetic field magnitude)

b. Length: $\bar{x} = \frac{x}{a}$

c. Wavenumber: $\bar{k} = ka$

d. Growth rate: $\bar{\gamma} = \gamma \tau_A$

e. Field Profile in x: i) $F(x) = \frac{B_{0y}(x)}{B_0}$

ii) $F'(x) = \frac{dF}{d\bar{x}}$, $F''(x) = \frac{d^2F}{d\bar{x}^2}$

5. Applying these transformations:

a. (A3) $$\boxed{\bar{\gamma}(\psi - F\phi) = S^{-1}\left(\frac{d^2}{d\bar{x}^2} - \bar{k}^2\right)\psi}$$

$$\overbrace{\frac{\partial B_x}{\partial t} - \frac{\partial}{\partial y}\left[U_x B_{0y}\right]}^{\text{Ideal Evolution}} = \overbrace{\frac{\eta}{\mu_0}\left(\nabla^2 B_x\right)}^{\text{Resistive term}}$$

b. (B3) $$\boxed{\bar{\gamma}^2\left(\frac{d^2}{d\bar{x}^2} - \bar{k}^2\right)\phi = -\bar{k}^2 F\left(\frac{d^2}{d\bar{x}^2} - \bar{k}^2 - \frac{F''}{F}\right)\psi}$$

$$\overbrace{\hat{z}\cdot \rho_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{U})}^{\text{inertia}} = \overbrace{\frac{\hat{z}\cdot\nabla\times[(\nabla\times\mathbf{B})\times\mathbf{B}_0] + \hat{z}\cdot\nabla\times[(\nabla\times\mathbf{B}_0)\times\mathbf{B}]}{\mu_0}}^{\mathbf{j}\times\mathbf{B}\text{ forces}}$$

E. Boundary Layer Problem:

1. Timescales of instability growth.

a. Slow relative to τ_A: $\gamma \ll \tau_A^{-1} \Rightarrow \gamma \tau_A = \boxed{\bar{\gamma} \ll 1}$

b. Fast relative to τ_R: $\gamma \gg \tau_R^{-1} \Rightarrow \gamma \tau_A \gg \frac{\tau_A}{\tau_R} = \frac{1}{S} \Rightarrow \boxed{\bar{\gamma} S \gg 1}$

2. Ordering in this limit:

a. i) $\underbrace{\psi - F\phi}_{\text{ii) Ideal } \mathcal{O}(1)} = \underbrace{\frac{1}{S\bar{\gamma}}\left(\frac{d^2}{d\bar{x}^2} - \bar{k}^2\right)\psi}_{\text{Resistive } \mathcal{O}\left(\frac{1}{S\bar{\gamma}}\right) \ll 1}$

iii) only contributes for small scales! $\Rightarrow$ boundary layer at $\bar{x} \simeq 0$.

iv) Outside of boundary layer: $\boxed{\text{Flux freezing}}$

(A4) $\boxed{\phi = \frac{\psi}{F}}$

b. i) $\underbrace{\bar{\gamma}^2\left(\frac{d^2}{d\bar{x}^2} - \bar{k}^2\right)\phi}_{\text{ii) Slow instability } \mathcal{O}(\bar{\gamma}^2) \ll 1} = \underbrace{-\bar{k}^2 F\left(\frac{d^2}{d\bar{x}^2} - \bar{k}^2 - \frac{F''}{F}\right)\psi}_{\text{Ideal } \mathcal{O}(1)}$

iii) On short timescales, $\boxed{\text{Force balance}}$ $\nabla \times (\mathbf{j} \times \mathbf{B}) = 0$

iv) (B4) $\boxed{\frac{d^2\psi}{d\bar{x}^2} - \bar{k}^2\psi - \frac{F''}{F}\psi = 0}$

3. a. Away from the boundary layer, $\left|\frac{x}{a}\right| \gg 1$,

equations (A4) & (B4) define frozen-in flux & force balance.

b. At $\bar{x} = \frac{x}{a} \to 0$, $\frac{\partial}{\partial \bar{x}} \gg 1$ and $\frac{\partial}{\partial \bar{x}} \gg \bar{k}$, so

we cannot neglect resistive and inertial terms

c. Note: At $\bar{x} = 0$, $F = \frac{B_{0y}(x)}{B_0} = 0$, so

equations (A4) & (B4) break down. $\Rightarrow$ $\boxed{\text{Include full equations}}$

4. Boundary Layer Equations:

a. In the limits $\bar{x} \ll 1$ and $\frac{d}{d\bar{x}} \gg 1$, we can Taylor expand $B_{0y}(x) = \cancel{B_{0y}(0)}^{0} + x \left.\frac{\partial B_{0y}}{\partial x}\right|_0 + \frac{x^2}{2} \left.\frac{\partial^2 B_{0y}}{\partial x^2}\right|_0 + \ldots$

b. Thus $F(x) = \frac{B_{0y}(x)}{B_0} = \bar{x} F'(0) + \frac{\bar{x}^2}{2} F''(0) + \ldots$

c. Renormalize variables:

i) Hydrodynamic Time $\boxed{\tau_H \equiv \frac{\tau_A}{k a F'(0)}}$

ii) Scalar function: $\tilde{\phi} = F'(0)\phi$

iii) Growth rate: $\tilde{\gamma} = \gamma \tau_H$

iv) Redefined Lundquist $\tilde{S} = \frac{\tau_R}{\tau_H}$

d. Applying this limit and renormalizations, we obtain

i) (A5) $\boxed{\tilde{\gamma}\left(\psi - \bar{x}\tilde{\phi}\right) = \frac{1}{\tilde{S}}\frac{d^2\psi}{d\bar{x}^2}}$

ii) (B5) $\boxed{\tilde{\gamma}^2 \frac{d^2\tilde{\phi}}{d\bar{x}^2} = -\bar{x}\frac{d^2\psi}{d\bar{x}^2}}$

F. Solution Strategy

1. Furth, Killeen, & Rosenbluth first developed this strategy to solve the boundary layer problem.

2. Strategy to solve Tearing Mode Stability problem

① Solve (A5) & (B5) at $|\bar{x}| \ll 1$ (Boundary Layer)

② Solve (A4) & (B4) everywhere else $|\bar{x}| \gtrsim 1$ (Ideal MHD)

③ Match two solutions at $\bar{x} = 0_+$ and $\bar{x} = 0_-$

④ Apply physical boundary conditions at $\bar{x} \to \pm\infty$

3. For a boundary layer of width $\Delta x = 2\delta$ centered at $x = 0$,

a. $\nabla \cdot \underset{\sim}{B} = 0$ implies that $\psi(x) = \frac{B_x(x)}{B_0}$ is continuous at $x = 0$!

b. Consider MHD solutions: $\psi_L(\bar{x}) \quad -\infty < \bar{x} < 0_-$

$\psi_R(\bar{x}) \quad 0_+ < \bar{x} < +\infty$

c. Thus continuity requires $\lim_{\delta \to 0} \left[\psi_R(+\delta) - \psi_L(-\delta) \right] = 0$

d. But, $\frac{d\psi}{d\bar{x}}$ need not be continuous at $\bar{x}=0$!

4. Def: Tearing Stability Index, Δ'

a. $$\boxed{\Delta' = \left[\frac{1}{\psi} \frac{d\psi}{d\bar{x}} \right]_{\bar{x}=0_-}^{\bar{x}=0_+}} \Rightarrow \boxed{\text{Unstable for } \Delta' > 0}$$

b. Depends on magnetic field equilibrium $B_{0y}(x)$, k, and boundary conditions at infinity.

c. For the Harris equilibrium $\underset{\sim}{B}_0 = B_0 \tanh\left(\frac{x}{a}\right)\hat{y}$,

$\psi(x)$ can be solved exactly to obtain

i) $$\boxed{\Delta' = \frac{1}{(ka)^2}\left[1-(ka)^2\right]}$$

ii) Unstable for $1-(ka)^2 > 0 \Rightarrow \boxed{ka < 1}$

d. We can solve for stability by simply solving the external (ideal MHD) equations, without solving for the internal (resistive MHD) boundary layer.

e. But, to determine the growth rate of the tearing instability, we need to solve the resistive layer equations.

G. Boundary Layer Solution

1. The boundary layer equations (A5) & (B5) can be solved with a somewhat complicated asymptotic analysis (pages of math). It involves
   a. Fourier transform of $\tilde{\phi}(\bar{x})$ and $\psi(\bar{x})$ in $\bar{x}$
   b. Matching of coefficients of asymptotic solution
   c. Assuming constant ψ approximation across layer

2. This procedure yields the result

a. $\Delta(\bar{\gamma},S) = 2\pi \dfrac{\Gamma(3/4)}{\Gamma(1/4)} \tilde{S}^{1/3} Q^{5/4}$

b. where $Q = \gamma \tau_H^{2/3} \tau_R^{1/3}$, $\tilde{S} = \dfrac{\tau_R}{\tau_H}$

c. Γ is the gamma function

3. Setting $\Delta(\bar{\gamma},S) = \Delta'$ and solving for γ yields the dimensional growth rate

a. $$\boxed{\gamma = \left[\frac{\Gamma(1/4)}{2\pi\Gamma(3/4)}\right]^{4/5} \frac{(\Delta')^{4/5}}{\tau_H^{2/5}\tau_R^{3/5}}}$$

b. This tearing mode is unstable if $\boxed{\Delta' > 0}$

c. Stable if $\Delta' < 0$

d. Hybrid timescale: $\tau_H^{2/5} \tau_R^{3/5}$

e. Valid for constant-ψ approximation, requiring $\Delta' \ll S^{1/3}$

I. G. (Continued) Howes (10)

4. Ignoring the constant in $[\;]^{4/5}$,

a. $\gamma \propto \frac{(\Delta')^{4/5}}{\tau_A} \frac{1}{\left(\frac{\tau_H}{\tau_A}\right)^{2/5}\left(\frac{\tau_R}{\tau_A}\right)^{3/5}} \propto \boxed{\frac{(\Delta')^{4/5}}{\tau_A}\left[ka\, F'(0)\right]^{2/5} S^{-3/5} \propto \gamma}$

larger Δ' (faster) — Smaller Scale (faster) — Steeper derivative (faster) — lower resistivity (slower)

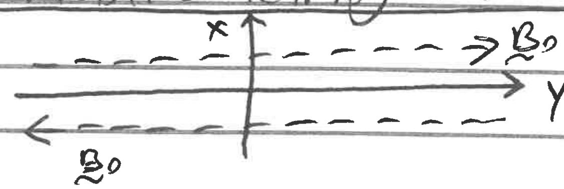
b. Hybrid timescale for tearing instability

$$\boxed{\tau_A \ll \frac{1}{\gamma} \ll \tau_R} \Rightarrow \frac{\tau_A}{\tau_A} \ll \frac{1}{\gamma\tau_A} \ll \frac{\tau_R}{\tau_A}$$

$$\mathcal{O}(1) \ll \mathcal{O}(S^{3/5}) \ll \mathcal{O}(S)$$

H. Summary: Linear Resistive Tearing Mode

1. $\underset{\sim}{B_0} = B_{0y}(x)\,\hat{y}$



2. Tearing Stability Index: $\Delta' \equiv \left[\frac{1}{\psi}\frac{d\psi}{dx}\right]_{0-}^{0+} = \frac{1-(ka)^2}{(ka)^2}$ ← for $B_{0y}(x) \propto \tanh\left(\frac{x}{a}\right)$

a. Unstable for $\Delta' > 0$, $ka < 1$

3. Growth Rate: $\gamma \propto (\Delta')^{4/5} S^{-3/5}$

4. Hybrid Timescale: a. $\underbrace{\tau_A}_{1} \ll \underbrace{\frac{1}{\gamma}}_{S^{3/5}} \ll \underbrace{\tau_R}_{S}$

b. $\mathcal{O}$ $\quad 1 \quad S^{3/5} \quad S$

c. Sweet-Parker (Forced) $\underbrace{\tau_A}_{1} \ll \underbrace{\tau_r}_{S^{1/2}} \ll \underbrace{\tau_R}_{S}$

$\mathcal{O}$ $\quad 1 \quad S^{1/2} \quad S$