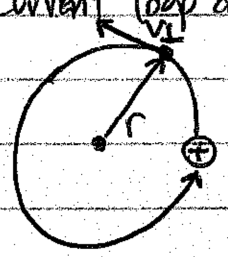
Lecture #3: Mirror Force, Adiabatic Invariance, Polarization Drift, and Collisions

I. Magnetic Moment and the Mirror Force

A. Magnetic Moment: 1. A current loop has magnetic moment $\mu = IA$

2. The current loop due to Larmor motion gives



$$\boxed{\mu = \frac{m v_\perp^2}{2B}}$$ Magnetic Moment

B. The Mirror Force: $\nabla B \parallel \underset{\sim}{B}$

1. Since $\nabla \cdot \underset{\sim}{B} = 0$ must always be satisfied, if the magnitude of B changes along the field line, it must involve a convergence of field lines:

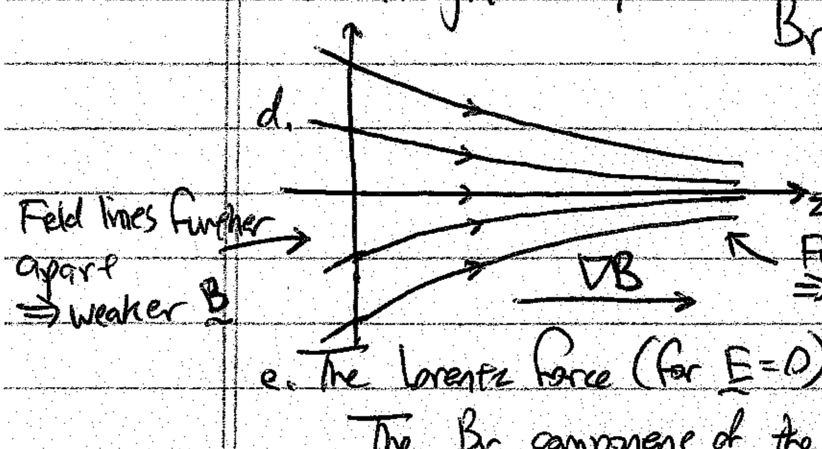
a. Consider an axisymmetric $\left(\frac{\partial}{\partial \phi} = 0\right)$ system in cylindrical coordinates:

$$\nabla \cdot \underset{\sim}{B} = \frac{1}{r}\frac{\partial}{\partial r}(r B_r) + \frac{\partial B_z}{\partial z} = 0$$

b. Thus, $$\frac{\partial}{\partial r}(r B_r) = -r \frac{\partial B_z}{\partial z}$$

c. Let us assume $\frac{\partial B_z}{\partial z}$ is independent of r (valid for small r), we can integrate to yield $$B_r = -\frac{r}{2}\frac{\partial B_z}{\partial z}$$

d.



Thus, increasing the field magnitude along z requires a B_r component.

e. The Lorentz force (for $\underset{\sim}{E} = 0$) is $\underset{\sim}{F} = q \underset{\sim}{v} \times \underset{\sim}{B}$.

The B_r component of the field crossed with v_ϕ leads to a force along the field (in the ~~z~~ direction).

Lecture #3 (Continued)

I. B. (Continued)

2. The Magnetic Mirror Force

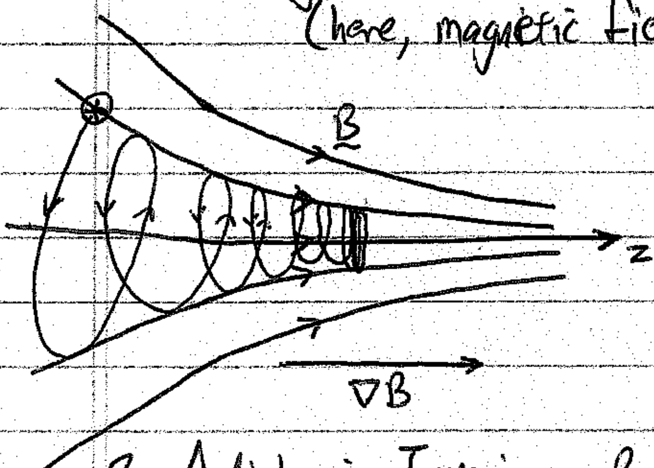
$$\boxed{F_z = -\mu \frac{\partial B_z}{\partial z}}$$

z is direction along the magnetic field.

a. Accelerates the particle along the field line in the direction of decreasing field magnitude

b. Force contains charged particles in weak $\underline{B}$-field regions

c. Analogous to the electrostatic force on a charge, $\underline{F} = -q \nabla \phi$. (here, magnetic field magnitude $|\underline{B}|$ behaves like a potential)



3. Adiabatic Invariance of Magnetic Moment

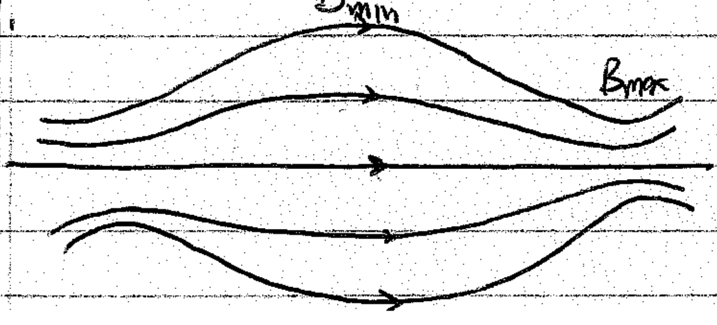
a. As a charged particle moves through a spatially or temporally varying $\underline{B}$ field, it can be shown that magnetic moment is conserved.

$$\boxed{\frac{d\mu}{dt} = 0}$$

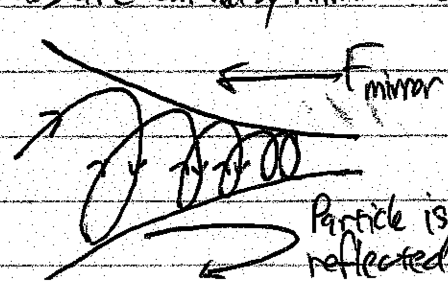
b. This invariance occurs when the change in the field occurs slowly compared to fast Larmor motion. ($\tau \ll \frac{1}{\Omega}$, or $r_L \ll L$)

C. Confinement by a Magnetic Mirror

1.



2. Particles are confined by mirror force



Lecture #3

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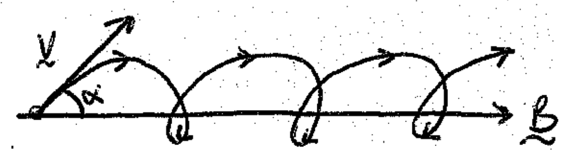
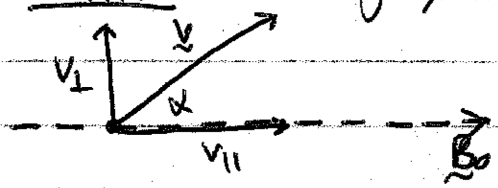
I.C. (Continued)



3. Pitch Angle

a. Definition: Pitch Angle, $\alpha \equiv$ angle between velocity vector & magnetic field.



$v_\parallel = v\cos\alpha$

$v_\perp = v\sin\alpha$

$\mathbf{v}\cdot\mathbf{B} = vB\cos\alpha$

4. Conservation of Energy: $\mathcal{E} = \frac{1}{2}mv_\parallel^2 + \frac{1}{2}mv_\perp^2 = \text{constant}$

a. Magnetic Moment $\mu = \frac{mv_\perp^2}{2B} = \text{constant}$

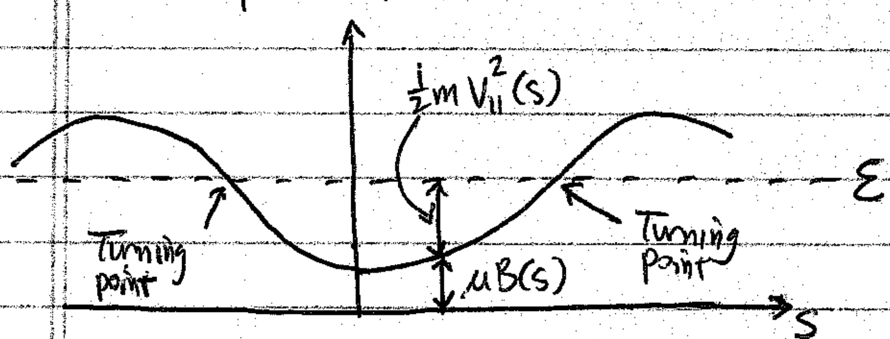
b. We can write $\frac{1}{2}mv_\perp^2 = \mu B$, so

Function of distance along magnetic field, "s"

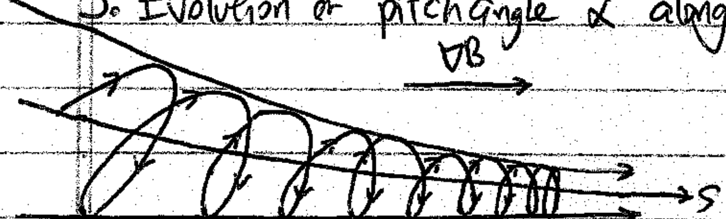
$$\boxed{\mathcal{E} = \frac{1}{2}mv_\parallel^2 + \mu B}$$

constant   constant

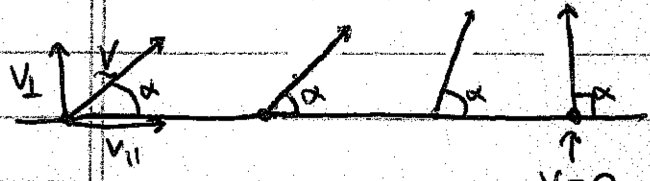
c. Energy Interpretation of Mirror Force



5. Evolution of pitch angle α along parallel coordinate s



a. $\mathcal{E} = \frac{1}{2}mv_\parallel^2 + \mu B = \text{constant}$



b. As $B(s)$ increases, conservation of energy implies $v_\parallel(s)$ decreases

c. Eventually, $v_\parallel(s) \to 0$, and particle turns around.

Lecture #3

I. (Continued)

D. Loss Cone

1. $\mathcal{E} = \frac{1}{2} m v^2 = \text{constant}$, so $v = \text{constant}$.

2. $$\mathcal{E} = \underbrace{\tfrac{1}{2} m v^2}_{=\mathcal{E}} \cos^2\alpha + \mu B \Rightarrow \mathcal{E}(1-\cos^2\alpha) = \mu B$$

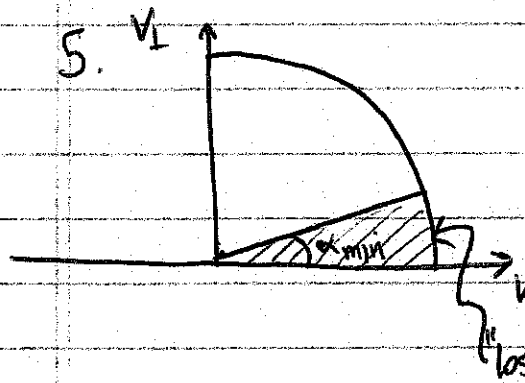
3. Thus, $$\boxed{\sin^2\alpha(s) = \frac{\mu B(s)}{\mathcal{E}}}$$

4. Particle is mirrored when $\alpha \to \frac{\pi}{2}$, or $\frac{\mu B(s)}{\mathcal{E}} \to 1$.

   a. But, there is a maximum value to $B(s)$, B_{max}.

   b. If $\mathcal{E} > \mu B_{max}$, then particle is never mirrored → it escapes through the "neck" of the magnetic bottle.

   c. But, the value of μ depends on $v_\perp$, or the pitch angle α.

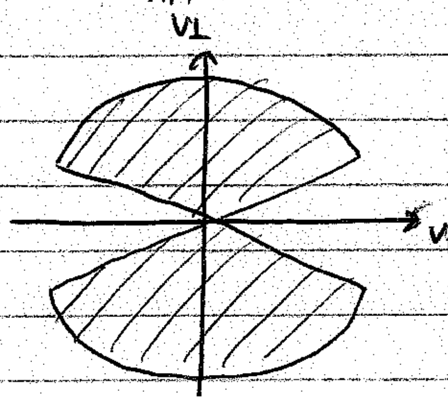
5. 

   a. Thus, particles with pitch angles below a threshold value will __always__ be lost.

   b. $$\boxed{\sin^2\underset{\substack{\uparrow \\ \text{loss cone angle.}}}{\alpha_{min}} = \frac{B_{min}}{B_{max}}}$$

   c. Particles with $\alpha < \alpha_{min}$ at B_{min} will be lost!

6. Loss Cone Distribution

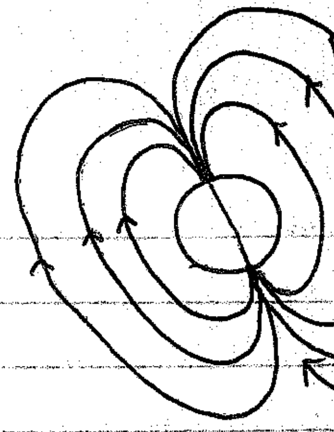


Particles confined in a magnetic mirror will lead to a "loss cone distribution."

Lecture #3 (Continued)

I. E. Magnetosphere:

1. The Dipole field is a natural magnetic mirror configuration



Region of weak $\underset{\sim}{B}$ field at equator

Regions of strong $\underset{\sim}{B}$ field at the poles

2. In the inner magnetosphere, one observes a loss cone distribution.

II. Adiabatic Invariance:

A. General Concepts

1. For any periodic motion, there exists a corresponding adiabatic invariant, related to the conservation of an action integral over the periodic motion from Hamiltonian mechanics.
2. In simple magnetic field configurations (magnetic mirror, magnetic dipole), there exist a hierarchy of characteristic periodic motions, leading to a hierarchy of adiabatic invariants.
3. The invariance occurs when changes to the system occur slowly (either temporally or spatially) compared to the fast periodic motion.

B. Periodic Motion and Adiabatic Invariants: Magnetic Mirror

Three types of periodic motion in an axisymmetric magnetic mirror:

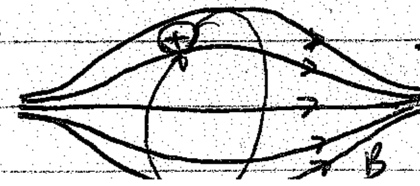
1. Larmor Motion:

1st: $\mu = \frac{m v_\perp^2}{2B}$

2. Parallel Bounce Motion: (Mirror Force)

2nd: $J = m \oint v_\parallel ds$

3. Azimuthal Drift Motion: (∇B & curvature drifts)



3rd: Magnetic Flux enclosed by drift orbit remains constant

Lecture #3 (Continued)

II. (Continued)

C. What types of periodic motion exist in a dipole magnetic field? (see HW)

III. Polarization Drift: Slowly varying $\underset{\sim}{E}$ and constant $\underset{\sim}{B}$

A. Multiple Timescale Analysis

1. Consider a case in which the timescale of the electric field variation $\tau \sim |E| / \left|\frac{\partial E}{\partial t}\right|$ is slow compared to Larmor motion, $\tau \ll \frac{1}{\Omega}$

2. We can solve for the motion, order by order, in terms of the small expansion parameter $\epsilon \sim \tau \Omega \ll 1$, to obtain

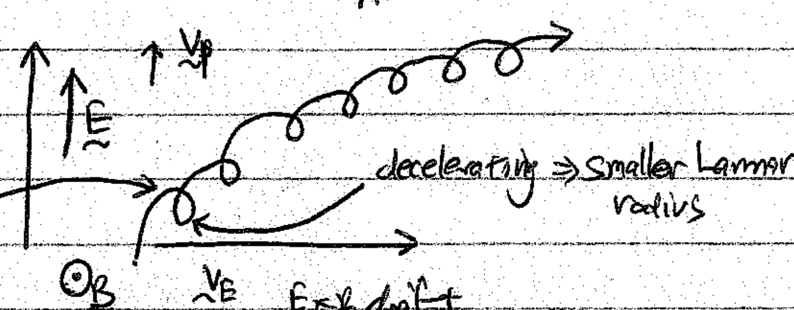
$$\underset{\sim}{v} = \underbrace{v_\perp \left[\cos(\Omega t)\,\hat{x} - \sin(\Omega t)\,\hat{y}\right]}_{\text{Zeroth-order Larmor Motion}} + \underbrace{\frac{\underset{\sim}{E}(t) \times \underset{\sim}{B}}{B_0^2}}_{\substack{\text{First-order}\\ \underset{\sim}{E}\times\underset{\sim}{B}\text{ drift}}} + \underbrace{\frac{1}{\Omega B_0}\frac{d\underset{\sim}{E}}{dt}}_{\substack{\text{Second-order}\\ \text{Polarization drift}}}$$

2. Define: Polarization Drift $\boxed{\underset{\sim}{v}_p \equiv \frac{1}{\Omega B}\frac{d\underset{\sim}{E}}{dt}}$

a. NOTE: $\frac{1}{\Omega B} = \frac{m}{qB^2}$, so polarization drift depends on charge

$\Rightarrow$ ions & electrons drift in opposite directions

3. Physical Picture: $\frac{dE}{dt} > 0$



4. Since electrostatic force is $\underset{\sim}{F} = q\underset{\sim}{E}$, and velocity is in direction of $\underset{\sim}{E}$, the $\underset{\sim}{E} \cdot \underset{\sim}{v} \neq 0 \Rightarrow$ Polarization drift changes particle energy.

IV. Collisions

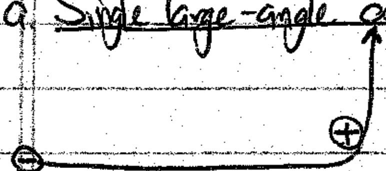
A. 1. Another physical process that affects the motion of a single charged particle is <u>collisions</u> with other charged particles.

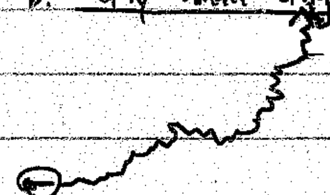
B. Single Large-Angle vs. Many Small-Angle Collisions

1. <u>Def</u>: <u>Collision time</u>, $\tau_c \equiv$ time required for particle trajectory to be deflected by $\pi/2$.

2. Deflection may be accomplished in different ways:

a. <u>Single large-angle collision</u> —OR— b. <u>Many small-angle collisions</u>





c. Because the Coulomb force is long-range, and many particles fall within the Debye sphere and can interact with the particle,

$\boxed{\text{Small-angle collisions dominate over large-angle collisions!}}$

d. It is important to remember that, within a single mean free path λ_m (defined as $\lambda_m = v_t \tau_c$), the particle actually experiences many small angle collisions.

C. Collision Frequency:

1. For a fully ionized hydrogenic (protons & electrons) plasma with density $n_0 = n_{0i} = n_{0e}$ and temperatures T_i and T_e,

$$\boxed{\nu_{e-i} = \frac{e^4}{2^{5/2}\pi\epsilon_0^2 m_e^{1/2}} \frac{n_0}{T_e^{3/2}} \ln N_D}$$

Collisions of electrons on ions

Plasma parameter, $N_D = \frac{4\pi}{3} n_0 \lambda_D^3$

IV. C. (Continued)

2. a. Note, because of the logarithmic, the dependence of the collision frequency on N_D is very weak
   b. Typically, $\ln N_D \sim 10-25$ for a wide range of plasmas.

3. The important dependencies are $\nu_{ei} \propto \frac{n_0}{T_e^{3/2}}$
   a. More density plasma is more collisional.
   b. Hotter plasma is less collisional → counterintuitive!
   c. Because space and astrophysical plasmas are usually very tenuous (low density) and hot, they are typically weakly collisional.

D. The "Fluid" limit of Plasmas

1. The strongly collisional limit, where λ_{mfp} is much smaller than all other scales of interest (typically system size L and Larmor radius r_L), corresponds to the usual fluid limit of hydrodynamics. (more on this in Lecture #4) $(\lambda_{mfp} \ll L)$
2. How can we possibly describe a weakly collisional plasma as a fluid?
   a. In a magnetized plasma, the motion of particles perpendicular to $\mathbf{B}$ is constrained by the Larmor radius ⇒ therefore, these perpendicular dynamics may be adequately described as a fluid.
   b. The mean free path relates only to motions parallel to $\mathbf{B}$.

E. Resistivity:
1. Like-particle collisions (e-e, i-i) conserve momentum, so resistivity (resistance to current flow) is determined by electron-ion collisions ν_{e-i}.