

Lecture #6 MHD Waves

I. Ideal MHD Equations

| | | |
|------------------------|---|--|
| Continuity Eq. | $\frac{\partial \rho}{\partial t} + (\underline{U} \cdot \nabla) \rho = -\rho(\nabla \cdot \underline{U})$ | <p>Compression</p> <p>Thermal Pressure Magnetic Pressure</p> <p>Magnetic Tension</p> <p>Stretching of Magnetic Field Lines</p> <p>$\nabla \cdot \underline{U}$ is Compression</p> |
| Momentum Eq. | $\frac{\partial \underline{U}}{\partial t} + (\underline{U} \cdot \nabla) \underline{U} = -\frac{1}{\rho} \nabla \left(p + \frac{B^2}{2\mu_0} \right) + \frac{(\underline{B} \cdot \nabla) \underline{B}}{\mu_0 \rho}$ | |
| Induction Eq. | $\frac{\partial \underline{B}}{\partial t} + (\underline{U} \cdot \nabla) \underline{B} = -\underline{B}(\nabla \cdot \underline{U}) + (\underline{B} \cdot \nabla) \underline{U}$ | |
| Adiabatic Eq. of State | $\frac{\partial p}{\partial t} + (\underline{U} \cdot \nabla) p = -\gamma p(\nabla \cdot \underline{U})$ | |

1. Here we have used the definition of the convective derivative

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \underline{U} \cdot \nabla$$

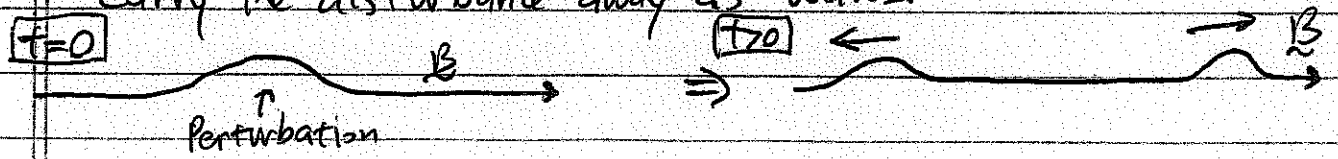
and the continuity equation to simplify the adiabatic eq. of state.

2. We have used vector identities [NRL p.4 (10)] and $\nabla \cdot \underline{B} = 0$ to simplify Induction eq.

II. Linear Wave Dispersion Relation

A. Linear Wave Modes

1. The linear wave modes of plasma describe the characteristic response of the plasma to small amplitude perturbations.
2. A general perturbation (of small amplitude) can be decomposed into its component linear MHD wave modes — those waves carry the disturbance away as waves.



II. A. (Continued)

3. Linear Dispersion Relation

- a. The linear dispersion contains all of the information about the characteristic linear wave modes.
- b. **IMPORTANT:** The technique for determining the linear dispersion relation arises frequently in the study of plasmas!

B. General Procedure for Calculating the Linear Dispersion Relation

1. Linearize the set of equations

2. Fourier transform in space and time

- a. Any disturbance can be written as a sum of Fourier modes:

$$\rho(x, t) = \sum_k \rho(k) e^{+i k \cdot x - i \omega(k) t}$$

Sum over all possible wave vectors $\rightarrow k$

↓
This frequency $\omega(k)$ is determined by the dispersion relation.

- b. Since equations are now linear, we need only solve for a single (arbitrary) wavevector k

- c. In practice, the Fourier transform simply replaces

$$\frac{\partial}{\partial t} \rightarrow -i\omega$$

$$\nabla \rightarrow i k$$

- d. These leads to an algebraic set of equations

3. Collect system of equations into $N \times N$ linear matrix equation.

$$\begin{pmatrix} N \times N \text{ matrix} \\ \approx \mathbf{D} \end{pmatrix} \begin{pmatrix} \end{pmatrix} = 0$$

$N \times 1$ vector of variables
($N=8$ for MHD: ρ, U, B, P)

II B₁ (Continued)

4. Solve for the determinant of $N \times N$ matrix = 0

$$|D| = 0 \leftarrow \text{Dispersion relation}$$

a. This leads to an expression $\omega = \omega(k)$
 ↑
 Solutions of dispersion relation.

C. Example of the Linear Dispersion Relation Calculation

1. Linearize: $\frac{\partial \underline{B}}{\partial t} + (\underline{U} \cdot \nabla) \underline{B} = -\underline{B} \nabla \cdot \underline{U} + (\underline{B} \cdot \nabla) \underline{U}$

a. Take uniform $\underline{B}_0 = B_0 \hat{z}$ in homogeneous plasma with no mean \underline{A}_w :

$$\left. \begin{aligned} \rho &= \rho_0 + \epsilon \rho_1 \\ \underline{U} &= \epsilon \underline{U}_1 \\ \underline{B} &= \underline{B}_0 + \epsilon \underline{B}_1 \\ p &= p_0 + \epsilon p_1 \end{aligned} \right\} \begin{aligned} &\text{where } \epsilon \ll 1 \text{ and} \\ &p_0, B_0, \text{ and } \rho_0 \text{ are constants} \\ &\text{in space and time} \end{aligned}$$

b. Substitute into equation

$$\frac{\partial \underline{B}_0}{\partial t} + \epsilon \frac{\partial \underline{B}_1}{\partial t} + \epsilon \underline{U}_1 \cdot \nabla \underline{B}_0 + \epsilon^2 \underline{U}_1 \cdot \nabla \underline{B}_1 = -\epsilon \underline{B}_0 \nabla \cdot \underline{U}_1 - \epsilon \underline{B}_1 \nabla \cdot \underline{U}_1 + \epsilon \underline{B}_0 \nabla \underline{U}_1 + \epsilon^2 \underline{B}_1 \nabla \underline{U}_1$$

c. Drop all terms with ϵ^2 or higher \Rightarrow

$$\boxed{\frac{\partial \underline{B}_1}{\partial t} = -\underline{B}_0 \nabla \cdot \underline{U}_1 + \underline{B}_0 \nabla \underline{U}_1} \leftarrow \text{Linearized Induction Eq.}$$

2. Fourier Transform: $\frac{\partial}{\partial t} \rightarrow -i\omega$ $\nabla \rightarrow i\vec{k}$

a. $-i\omega \underline{B}_1 = -i \underline{B}_0 \vec{k} \cdot \underline{U}_1 + i \underline{B}_0 \cdot \vec{k} \underline{U}_1$

b. $\boxed{\omega \underline{B}_1 = \underline{B}_0 (\vec{k} \cdot \underline{U}_1) - (\underline{B}_0 \cdot \vec{k}) \underline{U}_1}$

II. C2 (Continued)

c. Performing the Fourier Transform on all equations leads to

$$\begin{aligned} \textcircled{1} \quad \omega p_1 &= \rho_0 (\underline{k} \cdot \underline{U}_1) \\ \textcircled{2} \quad \omega \underline{U}_1 &= \underline{k} \left(\frac{p_1}{\rho_0} + \frac{\underline{B}_0 \cdot \underline{B}_1}{\mu_0 \rho_0} \right) - \frac{(\underline{B}_0 \cdot \underline{k}) \underline{B}_1}{\mu_0 \rho_0} \\ \textcircled{3} \quad \omega \underline{B}_1 &= \underline{B}_0 (\underline{k} \cdot \underline{U}_1) - (\underline{B}_0 \cdot \underline{k}) \underline{U}_1 \\ \textcircled{4} \quad \omega p_1 &= \gamma p_0 (\underline{k} \cdot \underline{U}_1) \end{aligned}$$

3. Collect system to $N \times N$ Matrix equation:

- In general, MHD has 8 unknowns ($\rho, U_x, U_y, U_z, B_x, B_y, B_z, p$)
- But, it is easier to compute the determinant of a 3×3 matrix
- Thus, eliminate $p_1, B_1,$ and ρ_1

- Use $\textcircled{4}$ to eliminate p_1

- Use $\textcircled{3}$ to eliminate B_1

- Since ρ_1 doesn't appear in other equations, no need to do anything.

d. Without loss of generality, we take $\underline{B}_0 = B_0 \hat{z}$

and $\underline{k} = k_\perp \hat{x} + k_\parallel \hat{z} = k \sin \theta \hat{x} + k \cos \theta \hat{z}$

e. Finally, we obtain the 3×3 Matrix equation for $(U_x, U_y, U_z) = \underline{U}_1$

$$\begin{pmatrix} \omega^2 - k^2 \sin^2 \theta (c_s^2 + v_A^2) - k \cos^2 \theta v_A^2 & 0 & -k^2 \sin \theta \cos \theta c_s^2 \\ 0 & \omega^2 - k^2 \cos^2 \theta v_A^2 & 0 \\ -k^2 \sin \theta \cos \theta c_s^2 & 0 & \omega^2 - k^2 \cos^2 \theta c_s^2 \end{pmatrix} \begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix} = 0$$

where we have defined:

Define: Alfvén Speed $v_A^2 \equiv \frac{B_0^2}{\mu_0 \rho_0}$

Define: Sound Speed $c_s^2 \equiv \frac{\gamma p_0}{\rho_0}$

Lecture #6 (Continued)

Pages 5

II. C. (Continued)

Not the same as the
3x3 Matrix \underline{D}

4. Solve for $|\underline{D}|=0$:

- At this step, we obtain the Dispersion Relation $D(\omega, k)=0$.
- We may then solve for the wave mode frequencies, $\omega = \omega(k)$.
- Computing the determinant and simplifying, we obtain:

$$(\omega^2 - k^2 \cos^2 \theta v_A^2) [\omega^4 - \omega^2 k^2 (c_s^2 + v_A^2) + k^4 \cos^2 \theta c_s^2 v_A^2] = 0$$

IDEAL MHD DISPERSION RELATION

III. Linear MHD Wave Properties

A. General: 1. Six Solutions to $D(\omega, k)$

- Two Alfvén waves
- Two Fast waves
- Two Slow waves

2. 3x3 Matrix Decouples into two systems:

$$(\omega^2 - k^2 \cos^2 \theta v_A^2) U_y = 0$$

Alfvén Waves

$$U_y \neq 0, U_x = U_z = 0$$

Typically most important
wave mode in astrophysical plasmas.

$$\begin{pmatrix} \omega^2 - k^2 \sin^2 \theta (c_s^2 + v_A^2) + k^2 \cos^2 \theta v_A^2 & -k^2 \sin \theta \cos \theta c_s^2 U_x \\ -k^2 \sin \theta \cos \theta c_s^2 & \omega^2 - k^2 \cos^2 \theta c_s^2 U_z \end{pmatrix} = 0$$

Fast and Slow Waves

$$U_x \neq 0, U_z \neq 0, U_y = 0$$

B. The Alfvén Waves

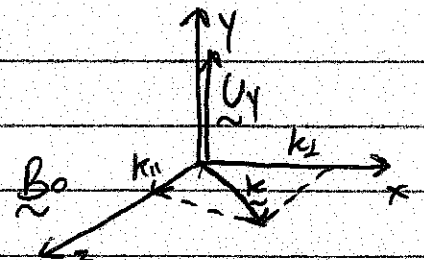
1. Solution:

$$\omega = \pm k_{\parallel} v_A$$

2. $U_y \neq 0 \Rightarrow$ Plasma fluid velocity out of plane containing k & B_0

3. $U_z = U_x = 0$

4. Therefore, $k \cdot U_1 = 0 \Rightarrow \nabla \cdot U_1 = 0 \Rightarrow$ Alfvén wave has no compression!



Lecture #6 (Continued)

III. B. (Continued)

5. What are the relevant linearized equations?

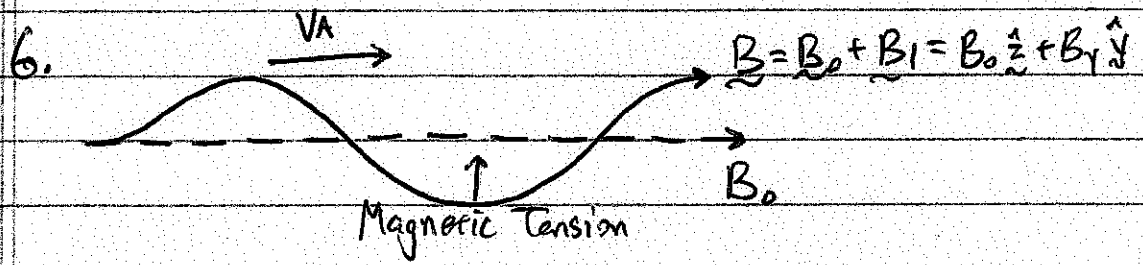
$$a. \frac{\partial U_y}{\partial t} = \frac{B_0}{\mu_0 \rho_0} \frac{\partial B_y}{\partial z}$$

$$b. \frac{\partial B_y}{\partial t} = B_0 \frac{\partial U_y}{\partial z}$$

Restoring Force is Magnetic Tension (Haves 6)

Wave equation: $\frac{\partial^2 U_y}{\partial t^2} = v_A^2 \frac{\partial^2 U_y}{\partial z^2}$

c. Since $\nabla \cdot \underline{U}_1 = 0$, then $\rho_1 = 0$ and $p_1 = 0$
 \Rightarrow No pressure or density fluctuations



- Alfvén wave is like a wave on a stretched rubber band.
- Wave propagates at speed v_A along magnetic field \underline{B}_0 .

C. Fast and Slow Waves

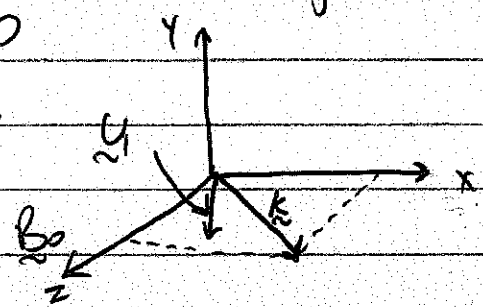
1. Solution:

$$\frac{c_s^2}{k^2} = \frac{1}{2}(c_s^2 + v_A^2) \pm \frac{1}{k} \sqrt{(c_s^2 + v_A^2)^2 - 4c_s^2 v_A^2 \cos^2 \theta}$$

⊕ sign: Fast wave

⊖ sign: Slow wave

2. $U_x \neq 0, U_z \neq 0, U_y = 0$
 Plasma Fluid velocity is in plane of \underline{k} & \underline{B}_0



3. $\underline{k} \cdot \underline{U}_1 \neq 0 \Rightarrow \nabla \cdot \underline{U}_1 \neq 0 \Rightarrow$ Fast & slow waves are compressible

4. Since $\nabla \cdot \underline{U}_1 \neq 0$, then $\frac{\partial \rho_1}{\partial t} \neq 0$ and $\frac{\partial p_1}{\partial t} \neq 0$, so

Fast and slow waves have density and pressure fluctuations

Lecture #6 (Continued)

III. C. (Continued)

Hoves ⑦

Greater "spring constant"
⇒ faster wave

5. Restoring Forces:

- a. Fast Wave:
1. Thermal and magnetic pressure act together
 2. Magnetic Tension
- b. Slow Wave:
1. Thermal and magnetic pressure oppose each other
 2. Magnetic Tension
- Lower "spring constant"
⇒ slower wave.

D. The Entropy Mode:

1. MHD Equations have 8 equations & 8 unknowns
⇒ Why do we only have 6 solutions?

2. Two Reasons:

a. $\nabla \cdot \mathbf{B} = 0$: This is an additional constraint, $k_1 B_x + k_2 B_z = 0$.
This eliminates one unknown, leaving 7 unknowns!

b. A more careful treatment allows fluctuations of ρ_1 and p_1 that satisfy specific entropy $s = c \frac{p}{\rho} = \text{const.}$

These entropy conserving fluctuations have $\omega = 0$, and have been dropped in our dispersion relation analysis.

3. So, there does exist a $\boxed{7^{\text{th}}, \omega = 0 \text{ Entropy Mode}}$ in addition to the 6 waves, (\neq Fast, \neq Alfvén, \neq Slow).

IV. More Properties of MHD

A. Dimensionless Dispersion Relation:

1. Define: $\bar{\omega} \equiv \frac{\omega}{kVA}$

Defines Plasma Beta $\beta = \frac{c_s^2}{VA^2}$

$$\boxed{(\bar{\omega}^2 - \cos^2 \theta) [\bar{\omega}^4 - \bar{\omega}^2(1 + \beta) + \beta \cos^2 \theta] = 0}$$

3. Demonstrates that the MHD Dispersion relation depends only on two parameters:

$$\boxed{\bar{\omega} = \bar{\omega}(\beta, \theta)}$$

Lecture #6 (Continued)

Hawes (8)

IV. (Continued)

B. Conservation of Energy in Ideal MHD:

1.

$$\frac{\partial}{\partial t} \left(\underbrace{\frac{1}{2} \rho U^2}_{\text{Kinetic Energy}} + \underbrace{\frac{P}{\gamma-1}}_{\text{Internal (Thermal) Energy}} + \underbrace{\frac{B^2}{2\mu_0}}_{\text{Magnetic Energy}} \right) + \nabla \cdot \left(\underbrace{\frac{1}{2} \rho U^2 \mathbf{U}}_{\text{Flux of Kinetic Energy}} + \underbrace{\frac{\partial P}{\partial t} \mathbf{U}}_{\text{Enthalpy Flux}} + \underbrace{\frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}}_{\text{Poynting Flux}} \right) = 0$$

2. Conserved Energy:
$$E = \frac{1}{2} \rho U^2 + \frac{P}{\gamma-1} + \frac{B_0^2}{2\mu_0}$$

C. MHD Linear Eigenfunctions:

1. Each wave mode has a particular eigenfunction (p_1, U_1, B_1, p_1)

a. Example: Alfvén wave: $p_1 = 0, p_1 = 0, U_x = 0, U_z = 0, B_x = 0, B_z = 0$

$$\frac{B_y}{B_0} = \pm \frac{U_y}{v_A}$$

2. How do we determine this?

a. Go back to 3×3 Matrix Equation for MHD Dispersion Relation.

b. Choose a value for one component, for example, $U_y = U_0$.

c. Solve for all other quantities.