Lectures on Turbulence

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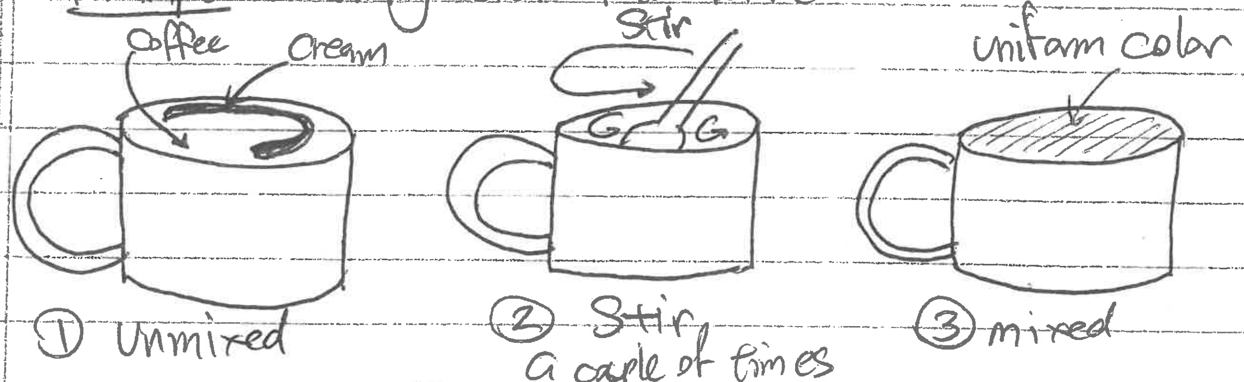
Nov 2009

Lecture #7: Isotropic Turbulence Theories: Hydrodynamic & MHD Turbulence

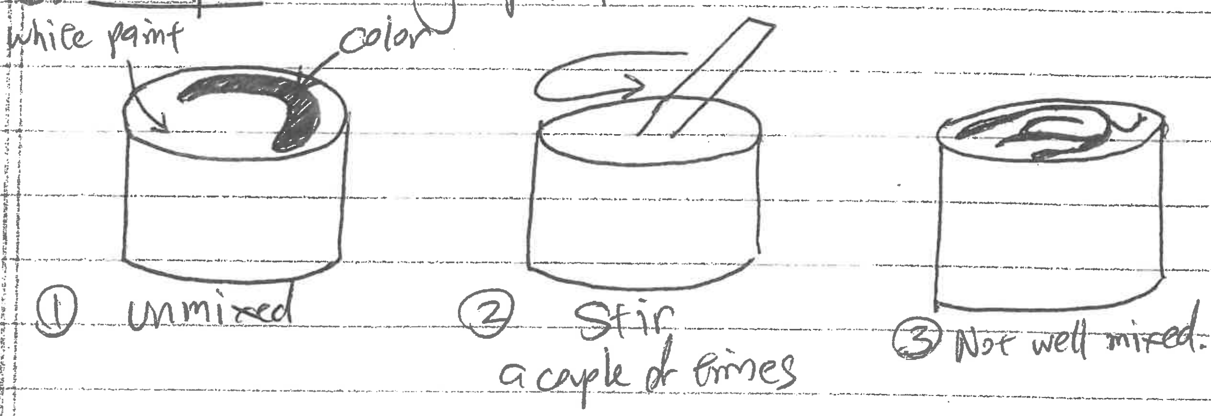
I. Phenomenological Picture of Turbulence

A. Example: Mixing Cream into Coffee



NOTE: With just a couple of short stirrings around the mug, the cream mixes rapidly to a smooth distribution.

B. Example: Mixing paint



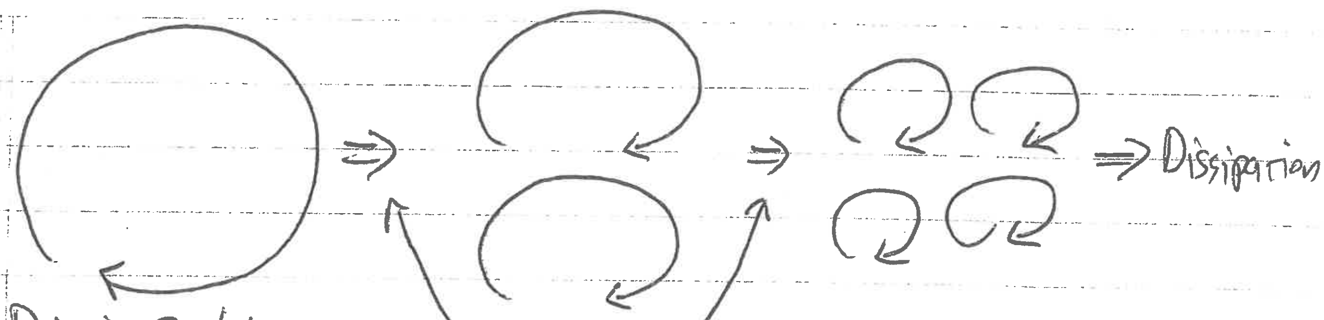
⇒ The difference in results is due to turbulence!
Flow in the coffee is turbulent, flow in the paint is laminar.

How can we describe the action of turbulence?

Lecture #7 (Continued)

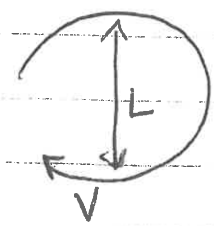
I. (Continued)

C. Cartoon Picture of Turbulence



1. Cascade continues until some dissipative mechanism damps the turbulence.
   a. In a hydrodynamic fluid (ie., water), viscosity damps the turbulence

2. Def: Reynold's Number: $\boxed{R \equiv \frac{LV}{\nu}}$



   a. ν = viscosity
   b. Turbulence flows occur when $R \gtrsim 100$.

II. Kolmogorov's Theory of Turbulence (1941)

Ref: Kolmogorov, A.N. (1941) The Local Structure of Turbulence in Incompressible Viscous Fluid for Very Large Reynolds Numbers, Dokl. Akad. Nauk SSSR, 30, 9.

Translation: (Proc. Roy. Soc. London, Ser A., 434, 9, 1991).

Lecture #7 (Continued)

II. (Continued)

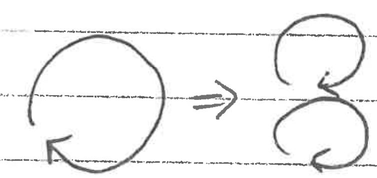
A. Kolmogorov's Hypothesis

1. The energy transfer in the inertial range is local in scale-space.
2. Energy cascade rate through inertial range is constant.

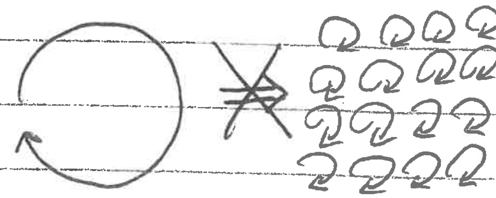
B. Local Energy Transfer

1. Transfer is dominated by interactions between eddies of similar scales:

a. $\ell \to \ell/2$



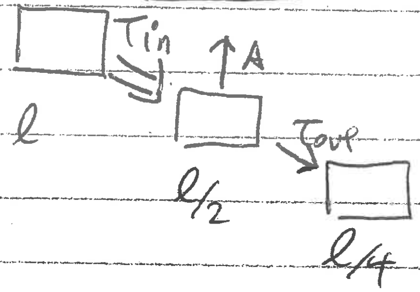
b. NOT $\ell \to \frac{\ell}{16}$



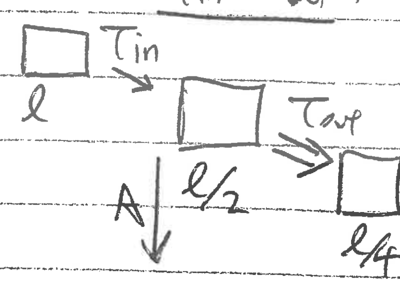
C. Constant Cascade Rate

1. In steady state, if cascade rate were not constant (a function of scale), energy would build up or diminish at some scale $\Rightarrow$ NOT a steady state

$T_{in} > T_{out}$:



$T_{in} < T_{out}$:



II. (Continued)

D. Energy Spectrum:

1. In general, a three dimensional fluctuation is a function of $\underset{\sim}{k} = (k_x, k_y, k_z)$, so the energy is given by

$$E(k_x, k_y, k_z)$$

2. This is the 3-dimensional energy spectrum $E^{(3)}(k_x, k_y, k_z)$.

a. Total Energy

$$E = \int dk_x \int dk_y \int dk_z \, E^{(3)}(k_x, k_y, k_z)$$

3. If the distribution of energy is isotropic in $\underset{\sim}{k}$-space, we can use spherical coordinates so that

$$E = \iiint k^2 dk \sin\theta \, d\theta \, d\phi \; E^{(3)}(k_x, k_y, k_z)$$

a. Integrating over θ and ϕ angles, we are left with

$$E = \int dk \; k^2 E^{(3)}(k) = \int dk \; E^{(1)}(k)$$

b. This defines the 1-dimensional energy spectrum $E^{(1)}(k)$

4. Turbulence literature can be confusing because some authors quote 3-D spectra, others 1-D spectra.

a. Ex: 3-D: $E^{(3)} \propto k^{-11/3}$

1-D: $E^{(1)} \propto k^{-5/3}$

b. Here we will discuss 1-D spectra only, so we'll drop the superscript "(1)".

c. 1-D Energy spectrum has units $\left[\frac{E}{k}\right]$

II. (Continued)

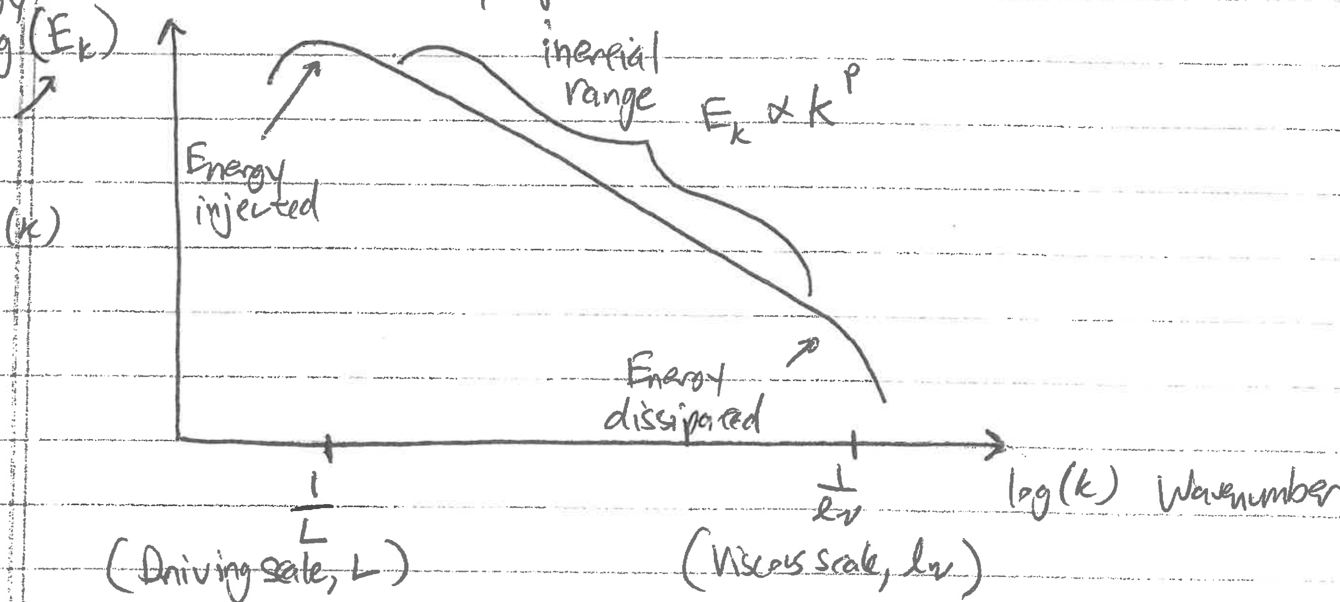
E. Inertial Range:

1. DEF: Inertial range: The range of scales in turbulence unaffected by driving or dissipation (bounded by the driving and dissipation scales). In this range, the physics is assumed to be self-similar.

2. Wavenumber Energy Spectrum

Kinetic Energy

Subscript notation $E_k = E(k)$



a. DEF: Spectral Index p: Slope of power-law (self-similar) spectrum on a log-log plot.

b. Self-similar physics yields power law behavior, giving a straight line on a log-log plot of E_k vs. k.

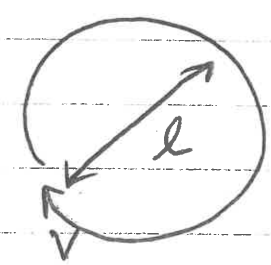
II (Continued)

F. Eddy Turnaround Time

1. The local hypothesis suggests the timescale τ for energy transfer at a given scale $\ell \sim \frac{1}{k}$ is determined only by the conditions at that scale.

2. This timescale in hydrodynamics is assumed to be the eddy turnaround time:

DEF: Eddy Turnaround Time: $\tau \sim \frac{\ell}{v}$



a. NOTE: In our scaling arguments, we drop all factors of order unity. Thus

$$\tau = \frac{\pi \ell}{v} \sim \frac{\ell}{v}$$

3. NOTE: In our scaling calculations, we will generally use wavenumber $k \propto \frac{1}{\ell}$ and frequency $\omega \propto \frac{1}{\tau}$.

a. Thus Eddy turnover frequency $\omega \sim kv$

b. Frequency of energy transfer from k to $2k$ is $\omega \sim k v_k$ $v_k = v(k)$

G. Kolmogorov's Scaling Arguments:

1. Assuming a constant energy transfer rate

$$\epsilon \sim \frac{(\text{energy})}{(\text{Time})} \sim (\text{energy})(\text{frequency}) \sim v_k^2 (k v_k)$$

a. Here we assume an incompressible fluid, $\rho = \rho_0$. Thus $\frac{E}{\text{Vol}} = \frac{1}{2}\rho_0 v^2 \sim v^2$

Lecture #7

II.G. (Continued)

2. Thus, a constant energy transfer rate gives

$$\epsilon_0 = \epsilon \sim k v_k^3$$

a. We can solve for $v_k = v(k)$, $\boxed{v_k = \epsilon_0^{\frac{1}{3}} k^{-\frac{1}{3}}}$

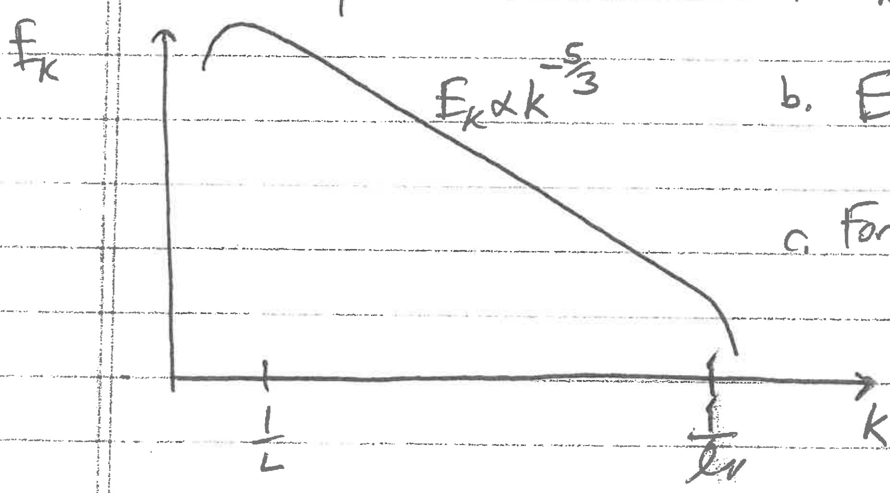
Thus $v_k \propto k^{-\frac{1}{3}}$

3. 1-D Energy Spectrum:

a. Energy spectrum $E_k = E(k) \sim \frac{\text{Energy}}{k} \sim \frac{v_k^2}{k} \sim \epsilon_0^{2/3} k^{-\frac{5}{3}}$

Therefore $\boxed{E_k \propto k^{-5/3}}$ This is the famous Kolmogorov Spectrum.

4. Summary:



a. $v_k \sim \epsilon_0^{\frac{1}{3}} k^{-\frac{1}{3}}$

b. $E_k \sim \epsilon_0^{2/3} k^{-5/3}$

c. For driving $v = v_0$ at $k \sim \frac{1}{L} = k_0$, $\epsilon_0 = k_0 v_0^3$, thus

$$E_k \sim \frac{v_0^2}{k_0} \left(\frac{k}{k_0}\right)^{-\frac{5}{3}}$$

d. NOTE: Cascade time $\tau_k \sim \frac{1}{\omega_k} \sim \frac{1}{k v_k} \sim \frac{1}{k \epsilon_0^{\frac{1}{3}} k^{-\frac{1}{3}}} \propto k^{-2/3}$

Since $\tau_k \propto k^{-2/3}$, cascade time gets shorter as turbulent motions go to smaller scale.

II. (Continued)

H. Return to Hydrodynamic Equations

1. The Euler Equations for Hydrodynamics are

Continuity Eq: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underset{\sim}{u}) = 0$

Force Eq: $\rho \left[\frac{\partial \underset{\sim}{u}}{\partial t} + \underset{\sim}{u} \cdot \nabla \underset{\sim}{u} \right] = -\nabla p$

Energy Eq: $\rho T \frac{ds}{dt} = 0$

where Specific energy $s = C_v \ln \left(\frac{p}{\rho^\gamma} \right)$

2. Focusing on the Force equation, $\frac{\partial u}{\partial t} + \underbrace{\underset{\sim}{u} \cdot \nabla \underset{\sim}{u}}_{\text{Nonlinear term.}} = -\frac{1}{\rho} \nabla p$

a. If we Fourier transform in ~~all~~ time and space, ~~the force eq.~~ becomes

$$\omega \underset{\sim}{u} - (\underset{\sim}{k} \cdot \underset{\sim}{u}) \underset{\sim}{u} = \frac{k p}{\rho}$$

linear frequency of evolution

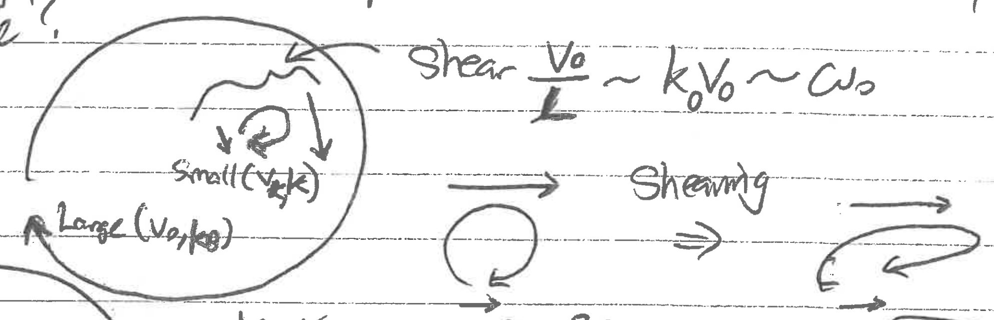
nonlinear frequency $\omega_{nl} = kv \rightarrow$ eddy turnaround frequency!

I. Testing the "Local Hypothesis"

1. Can big eddies shear apart small eddies before they cascade?

Shear $\frac{V_0}{L} \sim k_0 V_0 \sim \omega_0$

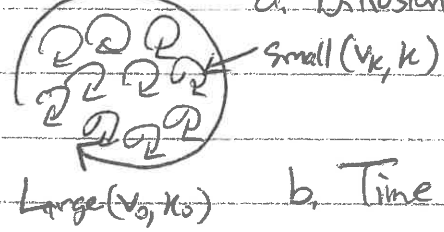
$V_k = V_0 \left(\frac{k_0}{k} \right)^{1/3}$



$$\frac{\omega_0}{\omega} = \frac{k_0 V_0}{k V_k} = \frac{k_0 V_0}{k \left[V_0 \left(\frac{k_0}{k} \right)^{1/3} \right]} = \left(\frac{k_0}{k} \right)^{2/3} \ll 1 \Rightarrow \boxed{\text{No!}}$$

II.I (Continued)

2. Can small eddies diffuse the fluid across a larger eddy in a big eddy cascade time?



a. Diffusion coefficient due to eddy of size $\ell \sim \frac{1}{k}$

is $D_k = \frac{\ell^2}{\tau_\ell} = \frac{\omega_k}{k^2} = \frac{k v_k}{k^2} = \frac{v_k}{k}$

b. Time to diffuse a distance $L \sim \frac{1}{k_0}$

$$\tau_D = \frac{L^2}{D_k} = \frac{1}{k_0^2 D_k} \sim \frac{1}{\omega_D}$$

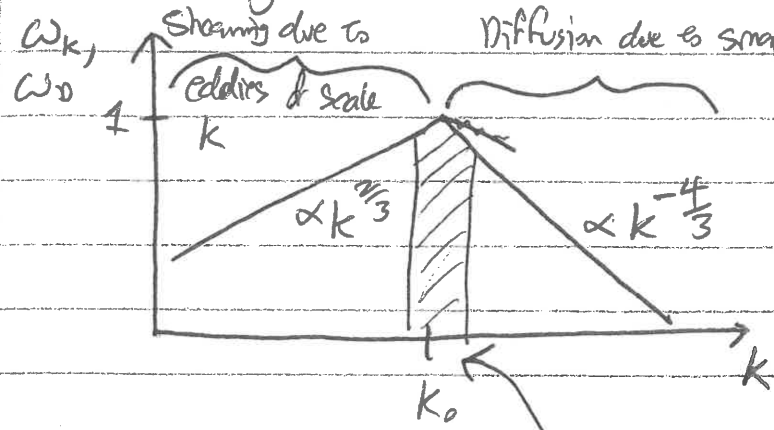
Thus $\omega_D \simeq k_0^2 D_k = k_0^2 \left(\frac{v_k}{k}\right) = k v_k \left(\frac{k_0}{k}\right)^2$

c. Compare ω_D to eddy frequency at large scale $\omega_0 = k_0 v_0$

$$\frac{\omega_D}{\omega_0} = \frac{k v_k}{k_0 v_0}\left(\frac{k_0}{k}\right)^2 = \frac{\cancel{v_0}\left(\frac{k_0}{k}\right)^{\frac{1}{3}}}{\cancel{v_0}}\left(\frac{k_0}{k}\right) = \left(\frac{k_0}{k}\right)^{\frac{4}{3}} \ll 1 \Rightarrow \boxed{\text{No!}}$$

$$v_k = v_0\left(\frac{k_0}{k}\right)^{\frac{1}{3}}$$

3. We can compare the frequency of energy transfer at a given scale k_0



"Local" energy transfer!

a. The peak occurs at $k \simeq k_0$. Thus, it is the shearing/diffusion due to the local scale eddies with $k \simeq k_0$ that dominates energy transfer from eddies at k_0 to smaller scales/higher wavenumbers, i.e. $k_0 \to 2k_0$

III. Iroshnikov-Kraichnan Theory of MHD Turbulence

Refs: 1. Iroshnikov, R.S. (1963) The Turbulence of a conducting fluid in a strong magnetic field, Astron. Zh. 40, 72 (Sov. Astron. 7, 566, 1964.)
2. Kraichnan, R.H. (1965) Inertial range spectrum of hydromagnetic turbulence, Phys. Fluids, 8, 1385.

A. How do we treat turbulence in a magnetized plasma?

1. Consider incompressible MHD:

$$\frac{\partial \underset{\sim}{v}}{\partial t} = -\underset{\sim}{v}\cdot\nabla\underset{\sim}{v} - \frac{1}{\rho_0}\nabla\left(p + \frac{B^2}{2}\right) + \frac{\underset{\sim}{B}\cdot\nabla\underset{\sim}{B}}{4\pi\rho_0}$$

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla\times(\underset{\sim}{v}\times\underset{\sim}{B})$$

$$\nabla\cdot\underset{\sim}{v}=0 \qquad \nabla\cdot\underset{\sim}{B}=0$$

2. Anisotropy: The presence of a magnetic field changes the dynamics in the directions perpendicular and parallel to the local magnetic field.

a. Kraichnan recognized that the presence of a mean magnetic field substantially changes the dynamics, unlike the presence of a mean flow.
   1. A mean flow can be transformed away.
   2. A mean magnetic field, however, introduces Alfven waves into the system that causes perturbations to be carried away along the magnetic field.

b. At small enough scales, the large scale field of a magnetized plasma looks like a mean field, and so MHD turbulence can be thought of as a collection of Alfven

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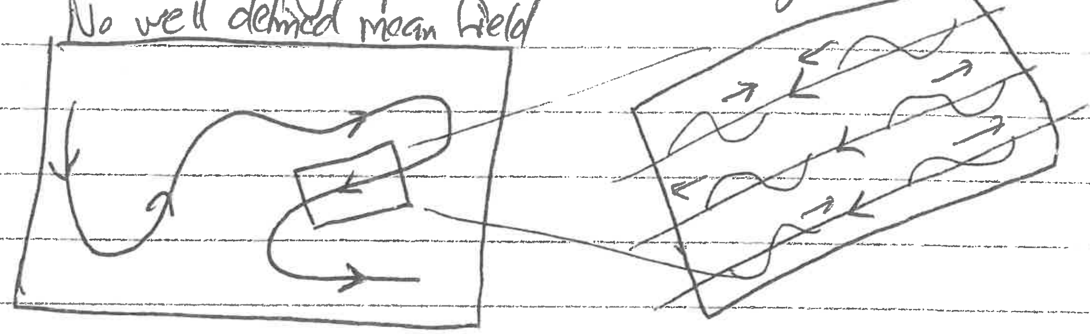
III. A.2 (Continued)

Waves travelling up and down the magnetic field.

No well defined mean field



3. Nonlinear Interactions:

a. The Incompressible MHD equations can be written in terms of Elsässer variables,

$$\underset{\sim}{z}^{\pm} = \underset{\sim}{v}_{\perp} \pm \frac{\delta \underset{\sim}{B}_{\perp}}{\sqrt{4\pi\rho_0}}$$

to give

$$\frac{\partial \underset{\sim}{z}^{\pm}}{\partial t} \underbrace{\mp (\underset{\sim}{V_A} \cdot \nabla) \underset{\sim}{z}^{\pm}}_{B_0 \text{ Linear Term}} + \underbrace{(\underset{\sim}{z}^{\mp} \cdot \nabla) \underset{\sim}{z}^{\pm}}_{\text{Nonlinear}} = -\nabla p \qquad (1)$$

where $\underset{\sim}{V_A} = \frac{\underset{\sim}{B_0}}{\sqrt{4\pi\rho_0}}$

b. $\underset{\sim}{z}^{+}$ represents a wave travelling down $\underset{\sim}{B_0}$

$\underset{\sim}{z}^{-}$ represents a wave travelling up $\underset{\sim}{B_0}$

c. NOTE: Nonlinear term $(\underset{\sim}{z}^{\mp} \cdot \nabla) \underset{\sim}{z}^{\pm}$ is non-zero only when both $\underset{\sim}{z}^{+} \neq 0$ and $\underset{\sim}{z}^{-} \neq 0$.

$\Rightarrow$ Nonlinear interactions occur only between Alfven waves traveling in opposite directions along the mean field

d. When $\underset{\sim}{z}^{-} = 0$, for example, eq (1) simplifies to $\boxed{\frac{\partial \underset{\sim}{z}^{+}}{\partial t} - \underset{\sim}{V_A} \cdot \nabla \underset{\sim}{z}^{+} = 0}$

III. A. 3. d (Continued)

- We can see this if we take $\nabla\cdot$ of eq(1):

$$\frac{\partial \nabla\cdot \underset{\sim}{z}^{+}}{\partial t} - \nabla\cdot[(\underset{\sim}{V_A}\cdot\nabla)\underset{\sim}{z}^{+}] = \nabla^2 p$$

(4) Since $\nabla\cdot\underset{\sim}{z}^{+}=0$ (for incompressible MHD), then $\nabla^2 p = 0$. Unless $p \not\to 0$ at ∞, we then see that pressure is constant everywhere, so $\nabla p = 0$, leaving our result

$$\frac{\partial \underset{\sim}{z}^{+}}{\partial t} - \underset{\sim}{V_A}\cdot\nabla\,\underset{\sim}{z}^{+} = 0$$

e. This describe a wave packet of arbitrary form and amplitude $\underset{\sim}{z}^{+}$ propagating in the $-\underset{\sim}{B_0}$ direction.

$\Rightarrow$ This is an <u>exact</u> nonlinear solution, true when all Alfven wave energy is moving in one direction.

f. Nonlinear interactions require $\underset{\sim}{z}^{-}\cdot\nabla\underset{\sim}{z}^{+}\neq 0$.

(1) In terms of Fourier modes, $\underset{\sim}{z}^{-}\cdot\underset{\sim}{k}^{+}\underset{\sim}{z}^{+}$,

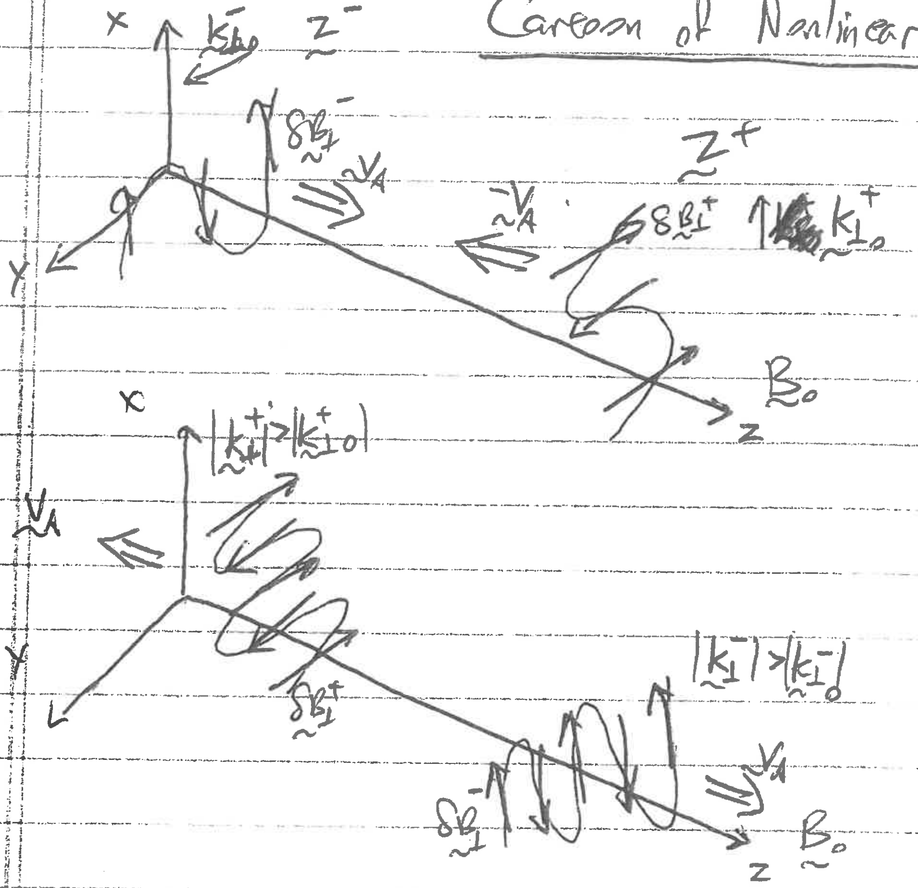
(2) Since $\underset{\sim}{z}^{-} = \underset{\sim}{V}_{\perp}^{-} - \frac{\delta B_{\perp}^{-}}{\sqrt{4\pi\rho_0}}$, it is in the perpendicular plane, so the <u>nonlinear frequency</u> $\boxed{\omega_{nl}\sim \underset{\sim}{k}_{\perp}^{+}\cdot\underset{\sim}{z}^{-}\sim k_{\perp}^{+}V^{-}}$

(3) Thus if $\underset{\sim}{k}_{\perp}^{+} = k_{\perp}\hat{\underset{\sim}{x}}$, then it is the $\hat{\underset{\sim}{x}}$ component of $\underset{\sim}{z}^{-}$ that leads to nonlinear evolution.

(4) For an Alfven wave with $\underset{\sim}{k}_{\perp}^{+} = k_{\perp}\hat{\underset{\sim}{x}}$, $\underset{\sim}{z}^{+} = z^{+}\hat{\underset{\sim}{y}}$

III. A. 3 f. (Continued)

Cartoon of Nonlinear Interaction



g. See video of Alfven Wave Collision Simulation, Verniero, Howes, & Klein, JPP 84: 905840103 (2018)

B. Iroshnikov-Kraichnan (IK) Spectrum

1. Assumes, like Kolmogorov's Hydrodynamic Theory, ~~that~~ the transfer of energy to small scales occurs isotropically in wave vector space.

2. a. Nonlinear interactions occur between Alfven wave packets of parallel length $l_\parallel \sim \frac{1}{k_\parallel} \sim \frac{1}{k}$

b. Interaction time is given by Alfven wave crossing time:

$$\tau \sim \frac{1}{k v_A} \sim \frac{1}{\omega}$$

c. Perturbation occuring in a single collision at wavenumber k:

$\frac{dv}{dt} \sim (v \cdot \nabla) v \sim \omega_{nl} v_k \sim (k v_k) v_k$

$$\boxed{\delta v_k \sim \frac{d v_k}{dt}\left(\frac{1}{k v_A}\right) \sim (k v_k) v_k \left(\frac{1}{k v_A}\right) \sim \frac{v_k^2}{v_A}}$$

III. B. 2 (continued)

d. Thus, the fractional perturbation in one collision is

$$\boxed{\frac{\delta v_k}{v_k} \sim \frac{v_k}{v_A}}$$

e. Assuming collisions are <u>uncorrelated</u>, each collision produces a random kicks. The number of random kicks needed to produce an order unity change $\delta v_k \sim v_k$, is

$$N \sim \left(\frac{v_k}{\delta v_k}\right)^2 \sim \left(\frac{v_A}{v_k}\right)^2$$

f. Therefore, the time to transfer all energy at k to $2k$ is

$$\tau \sim N\left(\frac{1}{k v_A}\right) \sim \left(\frac{v_A}{v_k}\right)^2 \frac{1}{k v_A} \sim \frac{v_A}{k v_k^2} \sim \frac{1}{\omega_k} \Rightarrow \boxed{\tau_k \sim \frac{v_A}{k v_k^2}}$$

g. So, assuming a constant energy cascade rate

$$\epsilon = \epsilon_0 = \frac{v_k^2}{\tau_k} = v_k^2 \omega_k = \frac{k v_k^4}{v_A}$$

(Compare to hydrodynamics, $\omega \sim k v_k^3$)

h. Therefore, for $\epsilon_0 = \frac{k_0 v_0^4}{v_A}$ for driving at k_0 with velocity v_0, we find

$$v_k = v_0\left(\frac{k}{k_0}\right)^{-\frac{1}{4}}, \text{ or } \boxed{v_k \propto k^{-\frac{1}{4}}}$$

3. 1-D Energy Spectrum: $E_k \sim \frac{v_k^2}{k} \sim v_0^2 k_0^{\frac{1}{2}} \left(\frac{k}{k_0}\right)^{-\frac{3}{2}}$, $\boxed{E_k \propto k^{-\frac{3}{2}}}$

This is referred to as the <u>Iroshnikov-Kraichnan Spectrum</u>.

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III. B. (Continued)

4. Weakening Turbulence:

a. Note the number of collisions N required at a given k:

$$N \sim \left(\frac{v_A}{v_k}\right)^2 \sim \frac{v_A^2}{v_0^2}\left(\frac{k}{k_0}\right)^{\frac{1}{2}} \quad \text{or} \quad \boxed{N \propto k^{\frac{1}{2}}}$$

The number of collisions required increases with k, so the nonlinear interactions become weaker at smaller scale.

b. Therefore, the theory formally has an infinite inertial range (until dissipation sets in).

IV. Summary: Isotropic Turbulence Theories

A. Hydrodynamic:

1. Kolmogorov Hypothesis a. Local Energy Transfer
b. Constant Energy Cascade rate
2. Nonlinear energy transfer frequency $\omega_k \sim k v_k$
3. 1-D Energy Spectrum $E_k \propto k^{-\frac{5}{3}}$

B. MHD:

1. MHD Turbulence can be thought of as Alfven waves travelling up and down a mean magnetic field
2. Nonlinear interactions occur only between oppositely directed Alfven waves.
3. IK Theory: a. Isotropic Energy Transfer
b. Nonlinear energy transfer frequency $\omega_k \sim \frac{k v_k^2}{v_A}$
c. 1-D Energy Spectrum $E_k \propto k^{-\frac{3}{2}}$