Lecture #8 Weak MHD Turbulence

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I. Review of Iroshnikov-Kraichnan (IK) MHD Turbulence Theory [Isotropic]

A. Incompressible MHD Equations

1. $$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = \frac{-1}{\rho_0}\nabla\left(p + \frac{B^2}{8\pi}\right) + \frac{\mathbf{B}\cdot\nabla\mathbf{B}}{4\pi\rho_0}$$

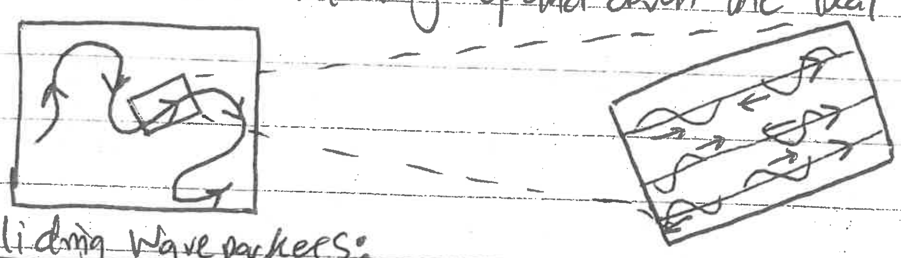
2. $$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{v}$$

3. a. $\nabla\cdot\mathbf{v} = 0$ b. $\nabla\cdot\mathbf{B} = 0$

B. IK Theory is based on four assumptions:

1. Kolmogorov Hypothesis: a. Local energy transfer in inertial range
   b. Energy cascade rate constant in inertial range

2. Kraichnan Hypothesis:
Even in a plasma without a well-defined mean magnetic field, at small scales MHD turbulence can be thought of as a collection of Alfvén waves traveling up and down the local mean field.



3. Colliding Wavepackets:
Nonlinear Interactions occur only between Alfven waves traveling in opposite directions along the mean magnetic field

Elsasser Form:
$$\frac{\partial \mathbf{z}^\pm}{\partial t} \mp \underbrace{(\mathbf{v}_A\cdot\nabla)\mathbf{z}^\pm}_{\text{Linear term}} + \underbrace{(\mathbf{z}^\mp\cdot\nabla)\mathbf{z}^\pm}_{\text{Nonlinear term}} = -\nabla p \quad , \quad \mathbf{z}^\pm = \mathbf{v}_\perp \pm \frac{\delta\mathbf{B}_\perp}{\sqrt{4\pi\rho_0}}$$

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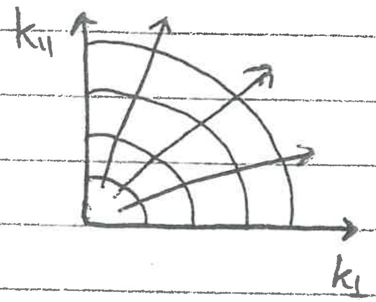
I. B. (Continued)

4. Isotropic transfer of energy in wave vector space:
This is an implicit assumption



Energy cascades isotropically to higher $k = \sqrt{k_\parallel^2 + k_\perp^2}$.

C. Predictions of IK Theory

1. $v_k = v_0 \left(\frac{k}{k_0}\right)^{-\frac{1}{4}}$

2. $E_k \propto k^{-\frac{3}{2}}$ 1-D Energy spectrum

3. Nonlinear interactions weaken as $k \to \infty$

$$N \sim \left(\frac{v_A}{v_k}\right)^2 \sim \frac{v_A^2}{v_0^2}\left(\frac{k}{k_0}\right)^{\frac{1}{2}} \propto k^{\frac{1}{2}}$$ → requires more collisions to cascade energy, so each collision is weaker

II. Weak MHD Turbulence [Anisotropic]

NOTE: The literature on weak MHD turbulence is immensely confusing because incorrect ideas were published, corrected only in later publications, but no review of this subject has been published that identifies the errors and states the currently accepted picture. The closest thing to a complete discussion is a long footnote in (Lithwick & Goldreich, 2003).
⇒ In the references section here I try to summarize the important papers and identify the errors briefly.

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II. (Continued)

A. Anisotropic Turbulence

1. Sridhar & Goldreich (1994) first pointed out that the IK theory assumes isotropy of the turbulent spectrum in wave vector space.

2. Also, they identified that the nonlinear interactions in IK theory are resonant 3-wave interactions.

$$\underset{\sim}{k}_1 + \underset{\sim}{k}_2 = \underset{\sim}{k}_3 \qquad \omega_1 + \omega_2 = \omega_3$$

B. Constraints on Resonant Interactions

Convention: $\omega > 0$, so sign of $k_\parallel$ indicates direction of propagation along $\underset{\sim}{B}_0$

1. Three-Wave Interactions: $\underset{\sim}{k}_1 + \underset{\sim}{k}_2 = \underset{\sim}{k}_3$ (1)

$\omega_1 + \omega_2 = \omega_3$ (2)

a. Note, for Alfven waves $\omega = |k_\parallel| v_A$, so (2) becomes

$$|k_{\parallel 1}| + |k_{\parallel 2}| = |k_\parallel|_3 \qquad (3)$$

b. Parallel component of (1) is $\qquad k_{\parallel 1} + k_{\parallel 2} = k_{\parallel 3}$ (4)

c. Case (1): $k_{\parallel 1}$ & $k_{\parallel 2}$ have the same sign

$\Rightarrow$ Both waves are propagating in the same direction, so nonlinear interaction is zero.

Case (2): $k_{\parallel 1}$ & $k_{\parallel 2}$ have opposite signs, take $k_{\parallel 1} \geq 0$, $k_{\parallel 2} \leq 0$.

$\Rightarrow$ For $k_{\parallel 2} \neq 0$, there is no solution to (3) & (4)!

d. Thus, $k_{\parallel 2} = 0$, so $k_{\parallel 1} = k_{\parallel 3}$, and there is no parallel cascade!

III. B. (Continued)

2. Four-Wave Interactions:

a. Point of confusion: Incompressible MHD has only quadratic nonlinearities, So how do we get 4-wave interactions?
Answer: The nature of the nonlinearities does not limit the order to which the equations can be solved perturbatively.

b. $\underset{\sim}{k}_1 + \underset{\sim}{k}_2 = \underset{\sim}{k}_3 + \underset{\sim}{k}_4 \xrightarrow{z\text{-component}} k_{\parallel 1} + k_{\parallel 2} = k_{\parallel 3} + k_{\parallel 4}$ (5)

$\omega_1 + \omega_2 = \omega_3 + \omega_4 \xrightarrow{\omega = |k_\parallel| v_A} |k_{\parallel 1}| + |k_{\parallel 2}| = |k_{\parallel 3}| + |k_{\parallel 4}|$ (6)

c. Again, $k_{\parallel 1}$ & $k_{\parallel 2}$ must have opposite signs for nonlinear term to be non-zero. Take $k_{\parallel 1} > 0$ & $k_{\parallel 2} < 0$.
Conservation laws imply $k_{\parallel 3}$ & $k_{\parallel 4}$ must also have opposite signs, so take $k_{\parallel 3} > 0$ $k_{\parallel 4} < 0$.

d. Thus (5) $\rightarrow$ $k_{\parallel 1} + k_{\parallel 2} = k_{\parallel 3} + k_{\parallel 4}$ (7)

(6) $\rightarrow$ $k_{\parallel 1} - k_{\parallel 2} = k_{\parallel 3} - k_{\parallel 4}$ (8)

e. The solution to (7) & (8) requires $k_{\parallel 1} = k_{\parallel 3}$ & $k_{\parallel 2} = k_{\parallel 4}$.
Again, there is no parallel cascade of energy.

3. Controversy about 3-wave vs. 4-wave interactions.

a. SMM83 first noted the 3-wave resonant interaction conditions and noted that either $k_{\parallel 1} = 0$ or $k_{\parallel 2} = 0$. This was used to explain numerical simulation results showing no parallel, only perpendicular, energy cascade.

b. SG94 argued 3-wave interactions were empty, and weak MHD turbulence must be based on 4-wave interactions.

c. MM95 & NB96 showed SG94 was wrong, 3-wave interactions non-zero

II. B. 3. (Continued)

d. GS97 derived the weak MHD Turbulence theory based on 3-wave interactions but said perturbation theory was inapplicable, so called it "intermediate" turbulence

e. GNNP00 showed perturbation theory can treat the three-wave interactions, so it really is "weak" turbulence. They suggested, however, that zero frequency ($k_{\parallel}=0$) modes may be problematic

f. LG03 discussed most of this controversy in detail, and explain that $k_{\parallel}=0$ modes are fine in perturbation theory as long as correlation times (parallel lengths) are shorter than cascade times. They explained the $k_{\parallel}=0$ modes correspond to field line wander

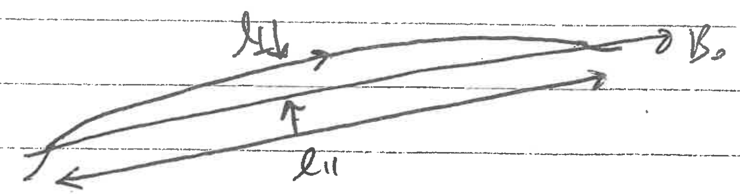
4. So what does all of this mean?

a. Three-wave interactions are dominant, and we will build a weak MHD Turbulence theory based on 3-wave interactions.

C. Scaling Theory for Anisotropic, Weak MHD Turbulence

1. Nonlinear interactions occur between oppositely directed Alfven wave packets.

a. Consider a wave packet with transverse scale $\ell_{\perp} \sim \frac{1}{k_{\perp}}$ and parallel scale $\ell_{\parallel} \sim \frac{1}{k_{\parallel}}$, where $\ell_{\parallel} \gg \ell_{\perp}$



II. C. 1. (Continued)

b. Perturbation occurring in a single collision at wavenumber $k_\perp$

$$\delta v_k \sim \frac{dv_k}{dt}\tau_{coll} \quad \text{where} \quad v_k \equiv v_\perp(k_\perp)$$

c. Collision time is Alfven wave crossing time: $\tau_{coll} \sim \frac{l_\parallel}{v_A} \sim \frac{1}{k_\parallel v_A}$

d. Nonlinear rate of change: $\frac{\partial \underset{\sim}{v}}{\partial t} \sim \underbrace{\underset{\sim}{v} \cdot \nabla}\, \underset{\sim}{v} \sim v_k k_\perp v_k$

For Alfven wave, $\underset{\sim}{v}$ is in $\perp$ direction.

e. Thus,

$$\delta v_k \sim \frac{k_\perp v_k^2}{k_\parallel v_A}, \quad \text{or fractional change} \quad \boxed{\frac{\delta v_k}{v_k} \sim \frac{k_\perp v_k}{k_\parallel v_A}}$$

2. Weak Turbulence: a. For weak turbulence $\frac{\delta v_k}{v_k} \ll 1$

b. Thus, to have an order unity change, $\delta v_k \sim v_k$, we need <u>many</u> collisions between Alfven waves.

c. But, successive Alfven waves will be <u>uncorrelated</u>, so each collision gives the velocity a random kick. The random walk will require

$$N \sim \left(\frac{v_k}{\delta v_k}\right)^2 \sim \left(\frac{k_\parallel v_A}{k_\perp v_k}\right)^2 \gg 1 \text{ collisions.}$$

3. Nonlinear Energy transfer time $\tau_{nl} \sim N\left(\frac{1}{k_\parallel v_A}\right) \sim \frac{(k_\parallel v_A)^2}{(k_\perp v_k)^2}\frac{1}{k_\parallel v_A} \sim \frac{1}{\omega_{nl}}$

$$\Rightarrow \boxed{\omega_{nl} \sim \left(\frac{k_\perp v_k}{k_\parallel v_A}\right) k_\perp v_k}$$

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II. C. (Continued)

4. DEF: Nonlinearity Parameter, $\boxed{\chi = \frac{k_\perp v_k}{k_\parallel v_A}}$

a. This is $\chi \sim \frac{\delta v_k}{v_k} \ll 1$ for weak turbulence $\Rightarrow$ Weak nonlinear interactions.

b. Note, therefore, $\omega_{nl} \sim \chi k_\perp v_k$

5. Energy Cascade Rate: $\epsilon \sim \frac{(\text{Energy})}{(\text{Time})}$ [redacted] $\sim (\text{Energy})(\text{Freq})$

a. $\epsilon \sim v_k^2 \left(\frac{k_\perp v_k}{k_\parallel v_A}\right) k_\perp v_k \sim \frac{k_\perp^2 v_k^4}{k_\parallel v_A} = \epsilon_0$

b. $v_k = \epsilon_0^{\frac{1}{4}} (k_\parallel v_A)^{\frac{1}{4}} k_\perp^{-\frac{1}{2}}$ $\qquad \boxed{v_k \propto k_\perp^{-\frac{1}{2}}}$

6. 1-D Energy Spectrum: $E_{k_\perp} \sim \frac{v_k^2}{k_\perp}$

a. NOTE: For Anisotropic spectra $E_{k_\perp} \equiv E(k_\perp)$, where

$$E = \int dk_\perp \, E(k_\perp) = \int dk_\perp \int_{-\infty}^{\infty} dk_\parallel \, 2\pi k_\perp E^{(3)}(\underset{\sim}{k}), \text{ so}$$

$$E(k_\perp) = \int_{-\infty}^{\infty} dk_\parallel \, 2\pi k_\perp E^{(3)} \underset{\sim}{k}$$

b. $E_{k_\perp} \sim \epsilon_0^{\frac{1}{2}} (k_\parallel v_A)^{\frac{1}{2}} k_\perp^{-2}$ $\Rightarrow \boxed{E_{k_\perp} \propto k_\perp^{-2}}$ Weak MHD Turbulence Spectrum

7. Nonlinear Interactions Become Increasingly Strong

a. $N \sim \left(\frac{k_\parallel v_A}{k_\perp v_k}\right)^2 \sim \left(\frac{k_\parallel v_A}{k_\perp \epsilon_0^{1/4} (k_\parallel v_A)^{1/4} k_\perp^{-\frac{1}{2}}}\right)^2 = \frac{(k_\parallel v_A)^{\frac{3}{2}}}{\epsilon_0^{\frac{1}{2}}} k_\perp^{-1} \propto k_\perp^{-1}$

The number on collisions required to achieve $\delta v_k \sim v_k$ decreases as $k_\perp$ increases. $\Rightarrow$ Each collision is stronger

II. C. 7. (Continued)

b. Nonlinearity Parameter $\chi \sim \frac{k_\perp v_k}{k_\parallel v_A}$ ← $k_\perp v_k \propto k_\perp^{\frac{1}{2}}$ → increases

← Denominator is constant → No $k_\parallel$ cascade.

1. In weak turbulence $\chi \ll 1$.

2. As $k_\perp$ increases, $\chi \to 1 \Rightarrow$ Nonlinear interactions get stronger,

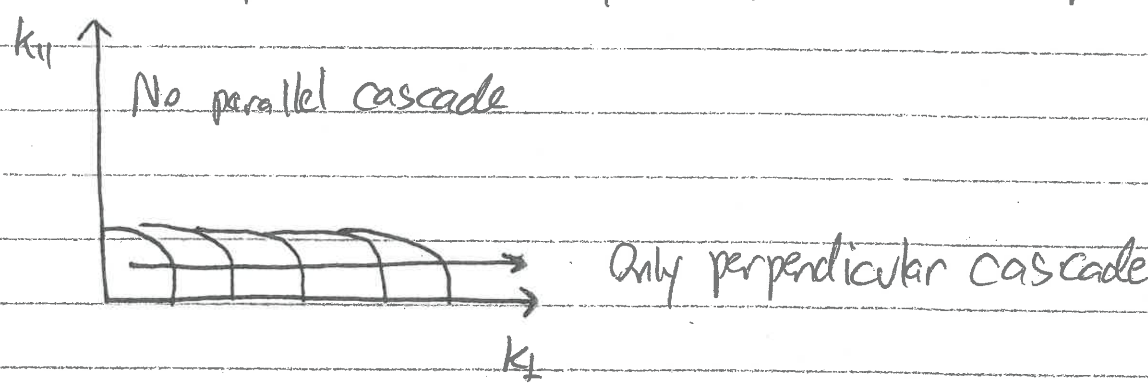
c. When $\chi \sim 1$, then $N \sim \frac{1}{\chi^2} \to 1$, all energy cascades in a single collision!

$\Rightarrow$ Turbulence is no longer weak, and perturbation theory fails

$\Rightarrow$ Transition to $\boxed{\text{Strong MHD Turbulence.}}$

D. Anisotropic Cascade

1. In a mean magnetic field B_0, fluctuations are statistically axisymmetric about the mean field, so wavevector space can be fully described by $(k_\perp, k_\parallel)$ 2-D plane.

2. 

III. Summary of Weak MHD Turbulence Theory

A. Major Differences with IK Theory

IK Theory	Weak MHD Turbulence
Isotropic	Anisotropic – No $k_\parallel$ cascade
$v_k \propto k^{-1/4}$	$v_{k_\perp} \propto k_\perp^{-1/2}$
$E_k \propto k^{-3/2}$	$E_{k_\perp} \propto k_\perp^{-2}$
$N \propto k^{1/2}$ – Weakening NL interactions	$N \propto k_\perp^{-1}$ – Strengthening NL interactions

B. Implications

1. Because $\chi \propto k_\perp^{1/2}$, as $k_\perp$ increases, eventually $\chi \to 1$.

$$\boxed{\text{All weak MHD turbulence strengthens as it cascades and eventually becomes strong MHD turbulence}}$$

2. $\chi \sim \dfrac{k_\perp v_\perp}{k_\parallel v_A}$ ← nonlinear / ← linear

$\chi \ll 1$ weak turbulence

$\chi \sim 1$ strong turbulence.

3. a. Therefore, weak MHD turbulence always has a limited range.

b. If turbulence is driven sufficiently strongly, range of scales in $k_\perp$ for which weak turbulence theory is valid may be small or nonexistent.

$$\boxed{\text{Next we turn to Strong MHD Turbulence}}$$

IV. References:

1. SMM83: Shebalin, JV, Matthaeus, WH, & Montgomery D. (1983) J. Plasma Physics 29, 525
   a. First discussed 3-wave resonant interactions requiring $k_\parallel = 0$ mode
2. SG94: Sridhar, S & Goldreich, P (1994) ApJ 432, 612.
   a. First anisotropic theory of weak MHD turbulence
   b. Incorrectly argued that 3-wave interactions were empty
   c. Based on 4-wave interaction (perturbation theory), 1D spectrum $E_{k_\perp} \propto k_\perp^{-7/3}$
3. MM95: Montgomery, D & Matthaeus, WH (1995) ApJ 447, 706.
   a. Short note claiming SG94 in error ~~of~~ 3-wave interactions not empty.
4. NB96: Ng, C.S. & Bhattacharjee, A (1996) ApJ 465, 845
   a. Presented detailed kinetic equations proving 3-wave interactions are not zero.
5. GS97: Goldreich, P & Sridhar, S (1997) ApJ 485, 680.
   a. Admitted 3-wave interactions, but said perturbation theory inapplicable
   b. Coined "Intermediate" turbulence, predicting $E_{k_\perp} \propto k_\perp^{-2}$
6. GNNP00: Galtier, S, Nazarenko, SV, Newell, AC, & Pouquet, A (2000) J. Plasma Phys. 63, 447.
   a. Showed perturbation theory is applicable to 3-wave interactions
   b. Very detailed derivation of kinetic equations
   c. First treated weak imbalanced MHD turbulence
7. LG03: Lithwick, Y & Goldreich P (2003) ApJ 582, 1220.
   a. Admits perturbation theory can treat 3-wave interactions. Thus "intermediate" turbulence is really weak turbulence.
   b. Paper aimed primarily at discussing imbalanced weak MHD turbulence
   c. Presents a short history to clarify all the controversy, but arguments are somewhat obscured by additional complications arising in imbalanced turbulence.