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TOWARD A THEORY OF INTERSTELLAR TURBULENCE. II. STRONG ALFVÉNIC TURBULENCE

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ABSTRACT

We continue to investigate the possibility that interstellar turbulence is caused by nonlinear interactions among shear Alfvén waves. Here, as in Paper I, we restrict attention to the symmetric case where the oppositely directed waves carry equal energy fluxes. This precludes application to the solar wind in which the outward flux significantly exceeds the ingoing one. All our detailed calculations are carried out for an incompressible magnetized fluid. In incompressible magnetohydrodynamics (MHD), nonlinear interactions only occur between oppositely direct waves. Paper I contains a detailed derivation of the inertial range spectrum for the weak turbulence of shear Alfvén waves. As energy cascades to high perpendicular wavenumbers, interactions become so strong that the assumption of weakness is no longer valid. Here, we present a theory for the strong turbulence of shear Alfvén waves. It has the following main characteristics. (1) The inertial-range energy spectrum exhibits a *critical balance* beween linear wave periods and nonlinear turnover timescales. (2) The "eddies" are elongated in the direction of the field on small spatial scales; the parallel and perpendicular components of the wave vector, k_z and $k_\perp$, are related by $k_z \approx k_\perp^{2/3} L^{-1/3}$, where L is the outer scale of the turbulence. (3) The "one-dimensional" energy spectrum is proportional to $k_\perp^{-5/3}$—an anisotropic Kolmogorov energy spectrum. Shear Alfvénic turbulence mixes specific entropy as a passive contaminant. This gives rise to an electron density power spectrum whose form mimics the energy spectrum of the turbulence. Radio wave scattering by these electron density fluctuations produces anisotropic scatter-broadened images. Damping by ion-neutral collisions restricts Alfvénic turbulence to highly ionized regions of the interstellar medium. We expect negligible generation of compressive MHD waves by shear Alfvén waves belonging to the critically balanced cascade. Viscous and collisionless damping are also unimportant in the interstellar medium (ISM). Our calculations support the general picture of interstellar turbulence advanced by Higdon.

Subject headings: ISM: general — MHD — turbulence

1. INTRODUCTION

This paper is concerned with strong turbulence due to nonlinear interactions among shear Alfvén waves in an incompressible, magnetized fluid. An earlier paper (Sridhar & Goldreich 1994, hereafter Paper I) offers a detailed analysis of *weak* shear Alfvénic turbulence. The principal results of Paper I are

1. Resonant 3-wave interactions among Alfvén waves are empty. The Iroshnikov-Kraichnan (Iroshnikov 1963; Kraichnan 1965) theory of incompressible MHD turbulence is based on resonant 3-wave interactions; consequently it is incorrect.
2. Resonant 4-wave interactions among *shear* Alfvén waves forbid transfer of energy to smaller spatial scales along directions parallel to the mean magnetic field. As energy cascades to large perpendicular wavenumbers, the 4-wave interactions strengthen; this limits the inertial range of the weak 4-wave energy spectrum since weak turbulence theory becomes inapplicable.

In § 2 we show that the strengthening of the weak 4-wave energy cascade requires inclusion of nonlinear "frequency renormalization" in the 4-wave kinetic equation. The renormalized kinetic equation allows for some transfer of energy to smaller spatial scales in the direction of the mean magnetic field which weakens the interactions. Based on arguments for both strengthening and weakening of the interactions, we propose that *strong* shear Alfvénic turbulence is a *critically balanced* cascade, and deduce the expected inertial-range energy spectrum. A kinetic equation for strong shear Alfvénic turbulence is derived in § 3. In § 4 we prove that the critically balanced energy spectrum is a stationary solution of this kinetic equation whose energy flux is directed towards large wavenumbers. Damping of shear Alfvén waves, due to collisions between ions and neutrals, by viscous damping, and by generation of other wave modes, is examined in § 5. The electron density fluctuation spectrum produced by turbulent mixing, and its implications for radio wave scattering are briefly covered in § 6. In § 7 we discuss applications to the real ISM, and the possible role of compressibility. We conclude in § 8 by relating our results to those obtained earlier by Montgomery & Turner (1981), Montgomery (1982), Montgomery, Brown, & Matthaeus (1987), and Higdon (1984).

2. STRONG ALFVÉNIC TURBULENCE; HEURISTICS

The equations governing the time evolution of an incompressible, magnetized fluid are

$$\begin{aligned} \partial_t \boldsymbol{b} &= \nabla \times (\boldsymbol{v} \times \boldsymbol{b}) + \kappa \nabla^2 \boldsymbol{b} \, , \\ \partial_t \boldsymbol{v} &= -(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{b} - \nabla p + \gamma \nabla^2 \boldsymbol{v} \, , \\ \nabla \cdot \boldsymbol{v} &= \nabla \cdot \boldsymbol{b} = 0 \, , \end{aligned} \tag{1}$$

where, $\boldsymbol{v}$ is the velocity field, $\boldsymbol{b} = \boldsymbol{B}/(4\pi\rho)^{1/2}$ is the magnetic field in velocity units, p is the ratio of the total (mechanical plus magnetic) pressure to the density, κ is the magnetic diffusivity, and γ is the kinematic viscosity. As in Paper I, we assume that the only role of magnetic diffusivity and viscosity is to provide a sink for energy on small spatial scales. In the considerations below, we drop these dissipative terms. The undisturbed fluid is in static equilibrium with a uniform magnetic field $\boldsymbol{B}_0 = B_0 \hat{\boldsymbol{z}}$. The Alfvén velocity $\boldsymbol{V}_{\rm A} = \boldsymbol{B}_0/(4\pi\rho)^{1/2}$. A weak excitation of shear Alfvén waves initiates a *weak* 4-wave cascade. As energy cascades to large perpendicular wavenumbers ($k_\perp$), the interactions between waves strengthen, ultimately invalidating the assumption of a *weak* cascade. The arguments are briefly summarized below:

Let v_λ be the perturbation in the amplitude of velocity and magnetic fields on parallel and perpendicular spatial scales $\lambda_\parallel \sim k_z^{-1}$ and $\lambda_\perp \sim k_\perp^{-1}$, respectively. Since we consider only shear Alfvén waves, the velocity and magnetic field perturbations are perpendicular to $\hat{\boldsymbol{z}}$. The strength of this perturbation can be characterized by a nonlinearity parameter

$$\zeta_\lambda \sim \frac{k_\perp v_\lambda}{k_z V_{\rm A}} \, . \tag{2}$$

For an isotropic excitation on spatial scale L, we have $k_\perp \sim k_z \sim L^{-1}$. If the excitation is weak, $v_L \ll V_{\rm A}$, implying that $\zeta_L \ll 1$. Such an excitation will undergo a *weak* 4-wave energy cascade to larger $k_\perp$, while k_z remains of the order of L^{-1}. The ratio of the cascade time to the wave period is

$$N \sim \zeta_\lambda^{-4} \sim \left(\frac{V_{\rm A}}{v_L}\right)^4 (k_\perp L)^{-4/3} \, . \tag{3}$$

The increase in ζ_λ as the cascade proceeds to larger $k_\perp$ is just a restatement of the growing strength of the interactions. When $\zeta_\lambda \sim 1$, the 4-wave perturbation analysis is invalid.

A small, but nonnegligible, ζ_λ is associated with a finite cascade time. We can include this effect in the kinetic equation (eqs. [28] and [29] of Paper I) by a nonlinear renormalization of the frequencies.[1] Viewed physically, the finite lifetimes of the waves give rise to finite line widths ("frequency-time uncertainty relationship"). So k_z tends to increase from its initially small value of L^{-1}; the renormalized interaction permits a transfer of energy to smaller spatial scales along the parallel direction. The fractional change in k_z per nonlinear interaction time is $\Delta k_z/k_z \sim N^{-1}$. As ζ_λ approaches unity from below, the growth rate of $k_z V_{\rm A}$ approaches that of $k_\perp v_\lambda$. Let us take stock of the arguments.

1. If ζ_λ is initially small, the excitation will undergo a *weak* 4-wave cascade, making ζ_λ grow in value.
2. Frequency renormalization allows for a transfer of energy to smaller spatial scales along the parallel direction.
3. As ζ_λ approaches unity from below, its growth ceases.

Each of these statements is based on weak, 4-wave couplings. Together, they lead us to conjecture that shear Alfvénic turbulence might achieve a state in which $\zeta_\lambda \sim 1$. In such a state, $k_z V_{\rm A} \sim k_\perp v_\lambda$; there is a balance between the Alfvén timescale (i.e., the linear wave period) and the intrinsically nonlinear timescale at which energy is transferred to shorter scales. In this sense, the energy cascade could be called a *critically balanced* cascade. We now estimate the energy spectrum of the critically balanced cascade.

Let energy be injected into the system roughly isotropically on spatial scale L. If the excitation is strong, $v_L \sim V_{\rm A}$, implying $\zeta_L \sim 1$. Then, the energy injected per unit mass, per unit time, on spatial scale L, is $\epsilon \sim V_{\rm A}^3/L$. Critical balance implies that the cascade time, $t_{\rm cas} \sim (k_z V_{\rm A})^{-1}$. Assuming a scale-independent cascade rate $\epsilon \sim v_\lambda^2/t_{\rm cas}$, we find that

$$k_z \sim k_\perp^{2/3} L^{-1/3} \, , \tag{4}$$

$$v_\lambda \sim V_{\rm A}(k_\perp L)^{-1/3} \, . \tag{5}$$

Equation (4) implies that the *parallel and perpendicular spatial sizes of eddies are correlated.* Since $k_\perp/k_z \sim (k_\perp L)^{1/3}$, as the cascade proceeds to larger $k_\perp$, the eddies become highly elongated along the direction of the magnetic field. Defining the three-dimensional energy spectrum, $E(k_\perp, k_z)$, by

$$\sum_{\rm eddies} v_\lambda^2 = \int d^3k E(k_\perp, k_z) \, , \tag{6}$$

we find that

$$E(k_\perp, k_z) \sim \frac{V_{\rm A}^2}{k_\perp^{10/3} L^{1/3}} \, f\!\left(\frac{k_z L^{1/3}}{k_\perp^{2/3}}\right) , \tag{7}$$

where $f(u)$ is a positive, symmetric function of u that becomes negligibly small when $|u| \gg 1$. We assume that f has unit height and width such that $\int_{-\infty}^{\infty} du f(u) \sim 1$.

[1] Usually this is a higher order effect, but since the 4-wave kinetic equation conserves wave frequencies, we must include renormalization.

3. A KINETIC EQUATION FOR STRONG ALFVÉNIC TURBULENCE

The dynamics of the critically balanced shear Alfvénic cascade is most conveniently formulated using Elsasser's (1950) variables. These may be defined as $\boldsymbol{U} = \boldsymbol{v} + \boldsymbol{b}$ and $\boldsymbol{W} = \boldsymbol{v} - \boldsymbol{b}$. Dropping the dissipative terms, we can rewrite equations (1) as

$$\begin{aligned} \partial_t \boldsymbol{U} &= -(\boldsymbol{W} \cdot \nabla)\boldsymbol{U} - \nabla p \,, \\ \partial_t \boldsymbol{W} &= -(\boldsymbol{U} \cdot \nabla)\boldsymbol{W} - \nabla p \,, \\ \nabla \cdot \boldsymbol{U} &= \nabla \cdot \boldsymbol{W} = 0 \,. \end{aligned} \tag{8}$$

In the undisturbed fluid (static, with a mean magnetic field), $\boldsymbol{U} = \boldsymbol{U}_0 = \boldsymbol{V}_A$ and $\boldsymbol{W} = \boldsymbol{W}_0 = -\boldsymbol{V}_A$. When the system is perturbed, $\boldsymbol{U} = \boldsymbol{U}_0 + \boldsymbol{u}$ and $\boldsymbol{W} = \boldsymbol{W}_0 + \boldsymbol{w}$, and equations (8) assume the form

$$\begin{aligned} \partial_t \boldsymbol{u} - V_A \partial_z \boldsymbol{u} &= -(\boldsymbol{w} \cdot \nabla)\boldsymbol{u} - \nabla p \,, \\ \partial_t \boldsymbol{w} + V_A \partial_z \boldsymbol{w} &= -(\boldsymbol{u} \cdot \nabla)\boldsymbol{w} - \nabla p \,. \end{aligned} \tag{9}$$

This is a particularly convenient way to write the dynamical equations of ideal, incompressible MHD: When $\boldsymbol{w} = 0$,

$$\partial_t \boldsymbol{u} - V_A \partial_z \boldsymbol{u} = -\nabla p \,. \tag{10}$$

Since $\nabla \cdot \boldsymbol{u} = 0$, $\nabla^2 p = 0$. The pressure at spatial infinity is uniform for localized disturbances; uniqueness of the solution to Laplace's equation implies that $\nabla p = 0$ everywhere. Therefore, $(\partial_t - V_A \partial_z)\boldsymbol{u} = 0$, with general solution $\boldsymbol{u} = \boldsymbol{u}(x, y, z + V_A t)$, describing a wave packet of arbitrary form traveling nondispersively in the negative z-direction. Similarly, when $\boldsymbol{u} = 0$, $\boldsymbol{w} = \boldsymbol{w}(x, y, z - V_A t)$ is a wave packet of arbitrary form traveling nondispersively in the positive z-direction. Nontrivial interactions occur only when there is overlap between the $\boldsymbol{u}$ and $\boldsymbol{w}$ wave packets. This led Kraichnan (1965) to conjecture that incompressible MHD turbulence occurs as a result of "collisions" between wave packets moving in opposite directions along the mean magnetic field. One of the few general statements that can be made is that these collisions are elastic; the wave packets emerge from a collision with the same energy they had before the collision. To see this, take dot products of equations (9) with $\boldsymbol{u}$ and $\boldsymbol{w}$, respectively. Incompressibility allows us to express the advective and pressure-gradient terms as total divergences. Hence only the time derivative terms survive an integral over all space giving

$$\frac{d}{dt}\int u^2\, d^3x = \frac{d}{dt}\int w^2\, d^3x = 0 \,. \tag{11}$$

The energy density is equal to $(v^2 + b^2)/2 = (u^2 + w^2)/4$. Since either u or w is zero in each of the wave packets at times much earlier to, or much after a collision, $u^2/4$ is indeed the energy density in the downward traveling wave packet, and $w^2/4$ is the energy density in the upward traveling wave packet. This remarkable conservation law is related (see e.g. Moffatt 1978) to the conservation (Woltjer 1958) of a topological invariant, namely, the *cross helicity* defined as

$$\begin{aligned} \mathscr{K} &= \tfrac{1}{2}\int (\boldsymbol{v} \cdot \boldsymbol{b}) d^3x \\ &= \tfrac{1}{8}\int (u^2 - w^2) d^3x \,. \end{aligned} \tag{12}$$

Therefore, separate conservation of the u and w energies is equivalent to the conservation of both the total energy and the cross-helicity. "Collision-induced" turbulence alters the shapes of oppositely moving wave packets while preserving their individual energies.

In equations (9), the role of p is (as in incompressible hydrodynamics) merely to ensure that $\boldsymbol{u}$ and $\boldsymbol{w}$ stay divergence free. It is straightforward to eliminate p by taking the divergence of equations (9), and working with the spatial Fourier transforms, $\tilde{\boldsymbol{u}}_k$ and $\tilde{\boldsymbol{w}}_k$, of $\boldsymbol{u}$ and $\boldsymbol{w}$, respectively. The Fourier transformed equations are

$$\begin{aligned} (\partial_t - i\omega_k)\tilde{\boldsymbol{u}}_k &= -\frac{i}{8\pi^3}\int d^3k_1\, d^3k_2 [\tilde{\boldsymbol{u}}_1 - \hat{\boldsymbol{k}}(\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{u}}_1)](\boldsymbol{k} \cdot \tilde{\boldsymbol{w}}_2)\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}) \,, \\ (\partial_t + i\omega_k)\tilde{\boldsymbol{w}}_k &= -\frac{i}{8\pi^3}\int d^3k_1\, d^3k_2 [\tilde{\boldsymbol{w}}_1 - \hat{\boldsymbol{k}}(\hat{\boldsymbol{k}} \cdot \tilde{\boldsymbol{w}}_1)](\boldsymbol{k} \cdot \tilde{\boldsymbol{u}}_2)\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}) \,, \end{aligned} \tag{13}$$

where caret (^) stands for a unit vector, $\omega_k = V_A k_z$ is the linear frequency of the shear Alfvén and the pseudo-Alfvén waves,[2] and $\tilde{\boldsymbol{u}}_1$ stands for $\tilde{\boldsymbol{u}}_{k_1}$ and so on. Both $\tilde{\boldsymbol{u}}_k$ and $\tilde{\boldsymbol{w}}_k$ are linear combinations of shear and pseudo-Alfvén waves. To study shear Alfvénic turbulence it is helpful to project $\tilde{\boldsymbol{u}}_k$ and $\tilde{\boldsymbol{w}}_k$ onto $\hat{\boldsymbol{e}}_k = \hat{\boldsymbol{k}}_\perp \times \hat{\boldsymbol{z}}$, the unit polarization vector of a shear Alfvén wave. The amplitudes, ϕ_k and ψ_k, of the upward and downward traveling shear Alfvén waves, respectively, are defined by

$$\tilde{\boldsymbol{w}}_k = i\phi_k \hat{\boldsymbol{e}}_k \,, \qquad \tilde{\boldsymbol{u}}_k = i\psi_k \hat{\boldsymbol{e}}_k \,. \tag{14}$$

[2] In Paper I, we defined $\omega_k = V_A |k_z|$ to be positive. In weak turbulence theory, interactions occur over many wave periods, with the result that ω_k can be interpreted as the "energy" of a "quasi-particle" in $\boldsymbol{k}$-space. For the critically balanced cascade we study in this paper, interactions are so strong that a "wave packet" lasts for at most a few wave periods; the "quasi-particle" interpretation is no longer useful. If we so wished, we could use only positive frequencies—the results are identical to those we arrive at.

Taking dot products of equations (13) with $-i\hat{e}_k$, we arrive at the following equations for the time evolution of the amplitudes:

$$(\partial_t - i\omega_k)\psi_k = \frac{1}{8\pi^3}\int d^3k_1\, d^3k_2\, \psi_1\phi_2(\hat{e}_k \cdot \hat{e}_1)(\boldsymbol{k} \cdot \hat{e}_2)\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}) ,$$
$$(\partial_t - i\omega_k)\phi_k = \frac{1}{8\pi^3}\int d^3k_1\, d^3k_2\, \phi_1\psi_2(\hat{e}_k \cdot \hat{e}_1)(\boldsymbol{k} \cdot \hat{e}_2)\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}) . \quad (15)$$

Next we derive the kinetic equation which governs the time evolution of the energy per mode in $\boldsymbol{k}$-space. The critically balanced spectrum (eq. [7]) emerges as a stationary solution which corresponds to a flux of energy to large $k_\perp$. To derive the kinetic equation, we begin with the energy density, $\mathscr{H}$ (in real space), defined by:

$$\mathscr{H} = \tfrac{1}{2}\langle v^2 + b^2\rangle = \tfrac{1}{4}\langle u^2 + w^2\rangle , \quad (16)$$

where angle brackets denote ensemble averages. Statistical spatial homogeneity allows us to define the power spectra, E_k^+ and E_k^-, of the amplitudes of the upward/downward traveling waves by

$$\langle \phi_k \phi_{k'}^*\rangle = 256\pi^6 E_k^+\, \delta(\boldsymbol{k} - \boldsymbol{k}') ,$$
$$\langle \psi_k \psi_{k'}^*\rangle = 256\pi^6 E_k^-\, \delta(\boldsymbol{k} - \boldsymbol{k}') . \quad (17)$$

The energy density $\mathscr{H} = \mathscr{H}^+ + \mathscr{H}^-$, where

$$\mathscr{H}^+ = \int d^3k E_k^+ \qquad \mathscr{H}^- = \int d^3k E_k^- , \quad (18)$$

from which it is clear that E_k^+ and E_k^- are the energy densities per unit volume in $\boldsymbol{k}$-space of upward/downward traveling waves, respectively; for brevity we call them simply "energy per mode". Both $\mathscr{H}^+$ and $\mathscr{H}^-$ are conserved energies. For future reference, we note that the cross helicity $\mathscr{K} = (\mathscr{H}^+ - \mathscr{H}^-)/2$.

We begin by working out $\partial_t\langle\phi_k\phi_{k'}^*\rangle$ and $\partial_t\langle\psi_k\psi_{k'}^*\rangle$

$$\partial_t\langle\phi_k\phi_{k'}^*\rangle = \langle\phi_{k'}^*\partial_t\phi_k\rangle + \langle\phi_k\partial_t\phi_{k'}^*\rangle ,$$
$$= i(\omega_{k'} - \omega_k)\langle\phi_k\phi_{k'}^*\rangle + \frac{1}{8\pi^3}\int d^3k_1\, d^3k_2\langle\phi_1\psi_2\,\phi_{k'}^*\rangle(\hat{e}_k \cdot \hat{e}_1)(\boldsymbol{k} \cdot \hat{e}_2)\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}) \quad (19)$$
$$+ \frac{1}{8\pi^3}\int d^3k_1\, d^3k_2\langle\phi_1^*\psi_2^*\,\phi_k\rangle(\hat{e}_{k'} \cdot \hat{e}_1)(\boldsymbol{k}' \cdot \hat{e}_2)\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}') .$$

Since $\partial_t\langle\phi_k\phi_{k'}^*\rangle$ is proportional (see eq. [17]) to $\delta(\boldsymbol{k} - \boldsymbol{k}')$, we have $\omega_k = \omega_{k'}$; hence the first term on the right side is zero. A third-order correlator appears in the integrands of the remaining two terms. The time evolution equation for this quantity involves a fourth-order correlator, and so on. Such an infinite hierarchy of equations occurs quite commonly in problems concerning turbulence and statistical physics. There really is no fundamental, yet tractable, solution to this problem. Some of the most popular "closure" approximations assume that the fourth-order correlator can be approximately expressed in terms of the second- and third-order correlators, thereby limiting the infinite hierarchy to just two equations—one each for the second- and third-order correlators. It is also a common procedure to further simplify, so as to arrive at just one equation for the second-order correlator.

We proceed by writing down the time evolution equation for the third-order correlator:

$$\partial_t\langle\phi_1\psi_2\,\phi_3^*\rangle = -i(\omega_1 - \omega_2 - \omega_3)\langle\phi_1\psi_2\,\phi_3^*\rangle + \frac{1}{8\pi^3}\int d^3k_4\, d^3k_5\{\delta(\boldsymbol{k}_4 + \boldsymbol{k}_5 - \boldsymbol{k}_1)\langle\psi_2\,\phi_3^*\,\phi_4\,\psi_5\rangle(\hat{e}_1 \cdot \hat{e}_4)(\boldsymbol{k}_1 \cdot \hat{e}_5)$$
$$+ \delta(\boldsymbol{k}_4 + \boldsymbol{k}_5 - \boldsymbol{k}_2)\langle\phi_1\phi_3^*\,\psi_4\,\phi_5\rangle(\hat{e}_2 \cdot \hat{e}_4)(\boldsymbol{k}_2 \cdot \hat{e}_5) + \delta(\boldsymbol{k}_4 + \boldsymbol{k}_5 - \boldsymbol{k}_3)\langle\phi_1\psi_2\,\phi_4^*\,\psi_5^*\rangle(\hat{e}_3 \cdot \hat{e}_4)(\boldsymbol{k}_3 \cdot \hat{e}_5)\} . \quad (20)$$

One of the simplest closures is the "quasi-normal approximation" (Millionshtchikov 1941), wherein the fourth-order correlators (occurring on the right side of eq. [20]) are treated as if ϕ_k and ψ_k are Gaussian random variables. This allows the fourth-order correlators to be expressed as sums of products of the second-order correlators (equivalently, the fourth-order cumulants are assumed to vanish). We assume that the excitation is such that, initially, there is no correlation between the upward and downward traveling waves. As time goes by, nonlinear interactions contribute to a nonzero, but small, value of $\langle\phi\psi\rangle$. To lowest order it is consistent to neglect such terms. Setting

$$\langle\psi_2\,\phi_3^*\,\phi_4\,\psi_5\rangle \simeq \langle\psi_2\,\psi_5\rangle\langle\phi_3^*\,\phi_4\rangle = (256\pi^6)^2 E_2^-\, E_3^+\, \delta(\boldsymbol{k}_2 + \boldsymbol{k}_5)\delta(\boldsymbol{k}_3 - \boldsymbol{k}_4) ,$$
$$\langle\phi_1\phi_3^*\,\psi_4\,\phi_5\rangle \simeq 0 , \quad (21)$$
$$\langle\phi_1\psi_2\,\phi_4^*\,\psi_5^*\rangle \simeq \langle\psi_2\,\psi_5^*\rangle\langle\phi_1\phi_4^*\rangle = (256\pi^6)^2 E_2^-\, E_1^+\, \delta(\boldsymbol{k}_2 - \boldsymbol{k}_5)\delta(\boldsymbol{k}_1 - \boldsymbol{k}_4) ,$$

we obtain the following approximate equation for the time evolution of the third-order correlator:

$$(\partial_t - 2i\omega_2)\langle\phi_1\psi_2\,\phi_3^*\rangle = \frac{(256\pi^6)^2}{8\pi^3}\,\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3)(\hat{e}_1 \cdot \hat{e}_3)(\boldsymbol{k}_3 \cdot \hat{e}_2)E_2^-(E_1^+ - E_3^+) , \quad (22)$$

where we have used $\delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3)$ to write $(\omega_1 - \omega_2 - \omega_3) = -2\omega_2$. Equations (17), (19), and (22) are equivalent to a single integro-differential equation for $E^+(\mathbf{k})$, which depends on $E^-(\mathbf{k})$ as well as $E^+(\mathbf{k})$. A similar procedure leads to an integro-differential equation for $E^-(\mathbf{k})$. For Navier-Stokes turbulence, the quasi-normal approximation does, indeed, reproduce the Kolmogorov inertial-range spectrum. But numerical integrations (Ogura 1963) of the "quasi-normally closed" equations (which include viscous damping that we neglect) of decaying turbulence develop negative energy spectra, pointing to a serious defect in the quasi-normal approximation. The physical reason (Orszag 1970) is that discarding fourth-order cumulants (similar to eq. [21]) results in an overestimate of the growth of the third-order correlators. The simplest, yet physical, remedy is to implement the "eddy-damped quasi-normal Markovian" (EDQNM) approximation (see Orszag 1977; Lesieur 1990).

Now we adapt the EDQNM closure to the problem at hand. The first step is to include a linear damping term in equation (22); on the left side, $-2i\omega_2$ is replaced by $-2i\omega_2 + \eta_2^-$, where $\eta^-(k)$ is an "eddy damping rate." Then, we integrate over time treating E^+ and E^- as time-independent quantities:

$$\langle \phi_1 \psi_2 \phi_3^* \rangle = \frac{(256\pi^6)^2}{8\pi^3}\, \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3)(\hat{e}_1 \cdot \hat{e}_3)(\mathbf{k}_3 \cdot \hat{e}_2) E_2^-(E_1^+ - E_3^+)\Theta_2^-(t) , \tag{23}$$

where

$$\Theta^-(\mathbf{k}, t) = \int_0^t ds \exp\left[(\eta_k - 2i\omega_k)(s - t)\right] = \frac{1 - \exp\left[(2i\omega_k - \eta_k)t\right]}{\eta_k - 2i\omega_k} . \tag{24}$$

Substituting equations (23) and (24) in equation (19), we obtain a kinetic equation for $E^+(\mathbf{k}, t)$. A similar derivation results in an equation for $E^-(\mathbf{k}, t)$.

$$\partial_t E_k^+ = 8 \int d^3k_1\, d^3k_2\, \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})(\hat{e}_1 \cdot \hat{e}_k)^2 (\mathbf{k} \cdot \hat{e}_2)^2 E_2^-(E_1^+ - E_k^+) \operatorname{Re}\left[\Theta_2^-(t)\right] . \tag{25}$$

$$\partial_t E_k^- = 8 \int d^3k_1\, d^3k_2\, \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})(\hat{e}_1 \cdot \hat{e}_k)^2 (\mathbf{k} \cdot \hat{e}_2)^2 E_2^+(E_1^- - E_k^-) \operatorname{Re}\left[\Theta_2^+(t)\right] . \tag{26}$$

Having derived kinetic equations for both upward and downward traveling shear Alfvén waves, a few comments are in order.

3.1. *Energy Conservation*

It may be verified that equations (25) and (26) conserve the energies of the upward and downward traveling waves, $\mathscr{H}^+$ and $\mathscr{H}^-$, defined in equation (18). By integrating over $\mathbf{k}$, we obtain

$$\partial_t \mathscr{H}^+ = 8 \int d^3k\, d^3k_1\, d^3k_2\, \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})(\hat{e}_1 \cdot \hat{e}_k)^2 (\mathbf{k} \cdot \hat{e}_2)^2 E_2^-(E_1^+ - E_k^+) \operatorname{Re}\left[\Theta_2^-(t)\right] . \tag{27}$$

Then interchanging $\mathbf{k}$ and $-\mathbf{k}_1$, and noting that $\mathbf{k} \cdot \hat{e}_2 = \mathbf{k}_1 \cdot \hat{e}_2$, we find $\partial_t \mathscr{H}^+ = -\partial_t \mathscr{H}^+$, and thus $\partial_t \mathscr{H}^+ = 0$. Therefore, both the total energy as well as the cross-helicity are conserved by the kinetic equations.

3.2. *Stationary, Symmetric Situations*

In quasi-stationary situations, we may set

$$\operatorname{Re}\left[\Theta_k^\pm(t)\right] \to \mathscr{T}_k^\pm = \operatorname{Re}\left[\Theta_k^\pm(\infty)\right] = \frac{\eta_k^\pm}{(\eta_k^\pm)^2 + (2\omega_k)^2} . \tag{28}$$

This expression, when substituted into the kinetic equations (25) and (26), accounts for the frequency renormalization discussed in § 2.

Because nonlinear interactions are limited to oppositely directed wave packets, each of the $\eta^\pm$ are properly viewed as functionals of both of the $E^\pm$. This peculiarity of Alfvénic turbulence restricts detailed application of the kinetic equations (25) and (26) to the symmetric case, for which $E_k^+ = E_k^- = E_k/2$ and the cross-helicity $\mathscr{K} = (E_k^+ - E_k^-)/2 = 0$. Henceforth, we restrict attention to stationary, symmetric solutions of the single kinetic equation

$$\partial_t E_k = 4 \int d^3k_1\, d^3k_2\, \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k})(\hat{e}_1 \cdot \hat{e}_k)^2 (\mathbf{k} \cdot \hat{e}_2)^2 E_2(E_1 - E_k)\mathscr{T}_2 , \tag{29}$$

where

$$\mathscr{T}_k = \frac{\eta_k}{\eta_k^2 + 4\omega_k^2} . \tag{30}$$

3.3. *Weak Turbulence*

When the wave amplitudes are small, the kinetic equation (29) describes resonant 3-wave interactions. For incompressible Alfvénic turbulence, such resonant interactions are null (Paper I). Let us see how this comes about. For consistency, the eddy damping coefficient, η_k, must be of order the rate at which energy is transferred into and out of mode $\mathbf{k}$. Having chosen η_k, we test for consistency by schematically carrying out the integrations over d^3k_1 and d^3k_2. In so doing we: treat the angular factors as being of order unity; take advantage of the local nature of the interactions in $\boldsymbol{k}$-space, and use a common value of $k_\perp$ for each of the three modes; express the energy density in an eddy having $\lambda_z \sim k_z^{-1}$ and $\lambda_\perp \sim k_\perp^{-1}$ in terms of the magnitude of its displacement vector ξ as

$$k_z k_\perp^2 E_k \sim \omega_k^2 \xi^2 . \tag{31}$$

Putting all these elements together, we arrive at

$$|\partial_t E_k| \lesssim \eta k_\perp^2 \xi^2 |E_1 - E_k| , \tag{32}$$

where we have made use of the inequality

$$\mathscr{T}_k < \frac{\eta}{\omega^2} . \tag{33}$$

To complete our estimate of $|\partial_t E_k|$, we note that weak turbulence requires $|\nabla\boldsymbol{\xi}| \ll 1$. Thus we arrive at the contradiction,

$$|\partial_t E_k| \ll \eta |E_1 - E_k| . \tag{34}$$

This confirms our contention that 3-wave resonant interactions are null.

3.4. *Strong Turbulence*

When the interactions are strong, the kinetic equations describe *nonresonant* 3-mode couplings. For investigating the inertial range of statistically steady, strong Alfvénic turbulence, it suffices to choose the eddy damping rate as

$$\eta(\boldsymbol{k}) = \eta_0 k_\perp^2 [|k_z| E(\boldsymbol{k}, t)]^{1/2} , \tag{35}$$

where η_0 is a dimensionless constant of order unity. This choice amounts to setting η equal to the eddy turn over rate or, in other words, to assuming that eddies of each scale are critically damped by turbulent viscosity associated with slightly smaller eddies. More general choices (Pouquet et al. 1975) are sometimes used for studying decaying turbulence, but we do not need them here.

Despite its deficiencies, the EDQNM closure provides a systematic procedure for deriving a kinetic equation for strong Alfvénic turbulence. The restriction to 3-wave interactions is not as great a limitation as it may seem, since interactions of *all* orders are expected to contribute with equal strengths to the critically balanced cascade. Thus, 3-wave interactions are an adequate proxy for those of all orders.

4. STATIONARY SOLUTIONS OF THE KINETIC EQUATION

Motivated by the energy spectrum (eq. [7]) for the critically balanced cascade, we look for stationary solutions of equation (29) having the form

$$E_k = A q^{-(\mu+\nu)} f(p/\Lambda q^\nu) , \tag{36}$$

where $\boldsymbol{q} = \boldsymbol{k}_\perp$, $p = k_z$, and A and Λ are positive constants. The eddy damping coefficient from equation (35) reads

$$\eta_k = \eta_0 q^2 [|p| E(\boldsymbol{q}, p)]^{1/2} . \tag{37}$$

The properties of the function f are defined following equation (7). We do not attempt to further determine its form. Instead, we set the more modest goal of seeking values of μ and ν that describe an inertial-range spectrum of strong shear Alfvénic turbulence. Since, for this purpose, the relevant information about f is its "width," Λq^ν in p for a fixed q, it suffices to deal with a kinetic equation for the dynamics in $\boldsymbol{q}$-space. To this end, we define a p-integrated energy density (energy per unit area in $\boldsymbol{q}$-space):

$$\mathscr{E}(q) = \int E_k \, dp = \frac{A\Lambda}{q^\mu} \tag{38}$$

Then integrating equation (29) over p, and noting that the dot products depend only on $\boldsymbol{q}$, $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$, we arrive at the "reduced" kinetic equation

$$\partial_t \mathscr{E}(q) = \int d^2q_1 \, d^2q_2 \, \delta(\boldsymbol{q}_1 + \boldsymbol{q}_2 - \boldsymbol{q})(\hat{e}_1 \cdot \hat{e}_q)^2 (\boldsymbol{q} \cdot \hat{e}_2)^2 (\mathscr{E}_1 - \mathscr{E}_q) G(q_2) . \tag{39}$$

The new function $G(q_2)$ results from the integral over p_2. It plays an essential role in determining the spectrum. We now derive explicit expressions for the dot products in the integrand and then for G:

1. *The dot products.*—Let θ_1 be the angle between $\boldsymbol{q}$ and $\boldsymbol{q}_1$, and θ_2 the angle between $\boldsymbol{q}$ and $\boldsymbol{q}_2$. Then

$$(\hat{e}_1 \cdot \hat{e}_q)^2 = \cos^2 \theta_1 .$$

We can use $\delta(\boldsymbol{q}_1 + \boldsymbol{q}_2 - \boldsymbol{q})$ to write

$$(\boldsymbol{q} \cdot \hat{e}_2)^2 = q^2 \sin^2 \theta_2 = \left(\frac{qq_1}{q_2}\right)^2 \sin^2 \theta_1 , \qquad q_2 = (q^2 + q_1^2 - 2qq_1 \cos \theta_1)^{1/2} .$$

2. *The function* $G(q)$.—In the computations to follow, we omit the subscript "2" on both q and p. Then G is defined as

$$G(q) = 4 \int dp\, \mathscr{T}(q, p) E(q, p) . \tag{40}$$

From equation (30),

$$\mathscr{T} = |2\omega|^{-1} \frac{|\eta/2\omega|}{1 + |\eta/2\omega|^2} ,$$

with

$$\left|\frac{\eta}{2\omega}\right| = \left(\frac{\eta_0 A^{1/2}}{2V_A \Lambda^{1/2}}\right) q^{(2-\mu/2-\nu)} \left|\frac{f(p/\Lambda q^\nu)}{p/\Lambda q^\nu}\right|^{1/2} .$$

Defining constants

$$\sigma_1 = \frac{\eta_0 A^{1/2}}{2V_A \Lambda^{1/2}} , \qquad \sigma_2 = \frac{2A\sigma_1}{V_A} , \tag{41}$$

we obtain the following expression for G:

$$G(q) = \sigma_2\, q^{(2-3\mu/2-2\nu)} \int_{-\infty}^{\infty} d\xi \left|\frac{f(\xi)}{\xi}\right|^{3/2} \left[1 + \sigma_1^2 \left|\frac{f(\xi)}{\xi}\right| q^{(4-\mu-2\nu)}\right]^{-1} .$$

This is the expression for $G(q)$ that appears in the reduced kinetic equation (eq. [39]). The other terms in the integrand of the collision integral are either angular factors, or pure power-law functions of q, q_1, or q_2. Therefore, for a stationary solution to equation (39), it is reasonable to expect $G(q)$ to be a power-law function of q. This is achieved when $(4 - \mu - 2\nu) = 0$, i.e., when

$$\nu = 2 - \mu/2 . \tag{42}$$

Using this relation between μ and ν we eliminate ν and write

$$G(q) = \frac{\sigma}{A\Lambda} q^{-(2+\mu/2)} , \tag{43}$$

where the new constant

$$\sigma = A\Lambda\sigma_2 \int_{-\infty}^{\infty} d\xi \left|\frac{f(\xi)}{\xi}\right|^{3/2} \left(1 + \sigma_1^2 \left|\frac{f(\xi)}{\xi}\right|\right)^{-1} . \tag{44}$$

With the above considerations in effect, the reduced kinetic equation (eq [39]) can be written as

$$\partial_t \mathscr{E}(q) = \sigma \int d^2q_1\, d^2q_2\, \delta(\boldsymbol{q}_1 + \boldsymbol{q}_2 - \boldsymbol{q}) \frac{q^2 q_1^2}{q_2^{(4+\mu/2)}} \cos^2 \theta_1 \sin^2 \theta_1 (q_1^{-\mu} - q^{-\mu}) . \tag{45}$$

Using the δ-function to eliminate q_2, and defining a new variable $r = q_1/q$,

$$\partial_t \mathscr{E}(q) = \sigma q^{(2-3\mu/2)} \int_0^{\infty} dr\, r^3 (r^{-\mu} - 1) \oint d\theta_1 \frac{\cos^2 \theta_1 \sin^2 \theta_1}{(1 + r^2 - 2r \cos \theta_1)^{(2+\mu/4)}} . \tag{46}$$

The transformation $r \to 1/r$ converts the integral over the range $(1, \infty)$ to an integral over the range $(0, 1)$. Thus

$$\partial_t \mathscr{E}(q) = \sigma q^{(2-3\mu/2)} \int_0^1 dr [r^3 - r^{(3\mu/2-1)}] (r^{-\mu} - 1) \oint d\theta_1 \frac{\cos^2 \theta_1 \sin^2 \theta_1}{(1 + r^2 - 2r \cos \theta_1)^{(2+\mu/4)}} . \tag{47}$$

The angular integral is positive definite. Hence it is obvious that the stationary solutions are given by

$$\mu = 8/3 , \quad \text{or} \quad \mu = 0 . \tag{48}$$

From the relation (eq. [42]) between μ and ν, we determine that $\mu = 8/3$ implies $\nu = 2/3$. This corresponds to the inertial-range spectrum of the critically balanced cascade (see eq. [7]). On the other hand, the $\mu = 0$ solution has equal energy per mode in $\boldsymbol{q}$-space; it corresponds to thermal equilibrium.

We now proceed to define the energy flux in $\boldsymbol{q}$-space, and then show that, as expected, the $\mu = 8/3$ solution carries a positive flux whereas the flux vanishes for the $\mu = 0$ solution. The energy density, $\mathscr{E}(q)$, and flux, $\hat{\boldsymbol{q}}\mathscr{F}$, are related by the continuity equation,

$$\partial_t \mathscr{E} + \nabla_q \cdot [\hat{\boldsymbol{q}}\mathscr{F}] = 0 . \tag{49}$$

From equation (45), the rate of change of energy density at $\boldsymbol{q}$ due to waves at $\boldsymbol{q}_1$ is

$$\dot{\mathscr{E}}(\boldsymbol{q}_1 \to \boldsymbol{q}) = \sigma \frac{q^2 q_1^2 \cos^2 \theta_1 \sin^2 \theta_1}{(q^2 + q_1^2 - 2qq_1 \cos \theta_1)^{(2+\mu/4)}} [q_1^{-\mu} - q^{-\mu}] . \tag{50}$$

The positive term arises from the creation of waves at $\boldsymbol{q}$ due to the coalescence of $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ waves. The negative contribution is due to the splitting of $\boldsymbol{q}$ waves into waves at $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. The energy flux

$$2\pi q \mathscr{F}(q) = \int_{q' > q} d^2 q' \int_{q_1 < q} d^2 q_1 \dot{\mathscr{E}}(\boldsymbol{q}_1 \to \boldsymbol{q}') . \tag{51}$$

From equations (50) and (51), it is evident that the $\mu = 0$ solutions have zero energy flux. We simplify the expression for the energy flux by using the detailed balance property of the energy transfer rate, $\dot{\mathscr{E}}(\boldsymbol{q}_1 \to \boldsymbol{q}) = -\dot{\mathscr{E}}(\boldsymbol{q} \to \boldsymbol{q}_1)$, which follows from the symmetry of the right-hand side of equation (50). Detailed balance implies

$$\int_{q' > q} d^2 q' \int_{q_1 > q} d^2 q_1 \dot{\mathscr{E}}(\boldsymbol{q}_1 \to \boldsymbol{q}') = 0 , \tag{52}$$

which when added to the right-hand side of equation (51) yields

$$2\pi q \mathscr{F}(q) = \int_{q' > q} d^2 q' \int d^2 q_1 \dot{\mathscr{E}}(\boldsymbol{q}_1 \to \boldsymbol{q}') . \tag{53}$$

Substituting equation (50) in equation (53), we arrive at an explicit formula for the energy flux:

$$2\pi q \mathscr{F}(q) = 2\pi\sigma \int_q^\infty dq' q'^{3(1-\mu/2)} \int_0^1 dr [r^3 - r^{(3\mu/2-1)}](r^{-\mu} - 1) \oint d\theta_1 \frac{\cos^2 \theta_1 \sin^2 \theta_1}{(1 + r^2 - 2r \cos \theta_1)^{(2+\mu/4)}} . \tag{54}$$

To check the sign of the energy flux corresponding to the $\mu = 8/3$ solution, we cannot simply set $\mu = 8/3$ in the expression on the right side of equation (54). If we did, then the q' integral would be equal to $\int_q^\infty dq'/q'$, giving a logarithmic ultraviolet divergence. On the other hand, the integrand of the r integral would vanish, so we would be left with an indeterminate $\infty \times 0$ form. Instead, we determine the sign of the flux by taking the limit $\mu \to 1$ from above. Then both the q' and r integrals are finite and *positive*, and so is the energy flux.

5. ESTIMATES OF DAMPING

We consider three damping mechanisms, each of which drains energy from shear Alfvénic turbulence.

5.1. *Ion-Neutral Damping*

Consider Alfvén waves in a partially ionized medium. The magnetic force acts directly on the ionized component, and indirectly on the neutral component through ion-neutral collisions. To the extent that the mean velocities of the two components differ, these collisions dissipate the energy of the Alfvén waves as heat. Let us estimate the damping rate of the critically balanced cascade in a partially ionized medium where the number density of ions is n_i, the number density of neutrals is n_0, and $n_0 \ll n_i$. We use the notation $v_{\rm th}$ for the (common) thermal speed of the ions and neutrals and assume that $v_{\rm th} \sim V_{\rm A}$ in what follows. The mean (over the thermal distribution) velocities of the ions and neutrals are denoted by $\boldsymbol{v}_i$ and $\boldsymbol{v}_0$, respectively. The neutrals have mean free path $l_0 \sim (n_i \sigma_0)^{-1}$, and collision frequency $\nu_0 \sim (v_{\rm th}/l_0)$, where σ_0 is the ion-neutral cross section. The different physical regimes for the collision process are characterized by the dimensionless parameters, Π_1 and Π_2:

$$\Pi_1 = l_0 k_\perp , \qquad \Pi_2 = \frac{\omega_k}{\nu_0} . \tag{55}$$

Deep in the inertial range of the critically balanced cascade, $k_\perp \gg k_z \sim k_\perp^{2/3} L^{-1/3}$. Hence

$$\frac{\Pi_1}{\Pi_2} = \left(\frac{v_{\rm th}}{V_{\rm A}}\right) \frac{k_\perp}{k_z} \sim \left(\frac{v_{\rm th}}{V_{\rm A}}\right) (k_\perp L)^{1/3} \gg 1 . \tag{56}$$

There are two limiting cases:

1. $\Pi_1 \ll 1$.—Equation (56) implies $\Pi_2 \ll 1$ as well. The neutrals move almost in step with the ions. The small mean slip velocity, $\boldsymbol{v}_{\rm slip} \equiv \boldsymbol{v}_0 - \boldsymbol{v}_i$, is of order

$$\boldsymbol{v}_{\rm slip} \sim -\frac{1}{\nu_0} \frac{d\boldsymbol{v}_i}{dt} - (\Pi_1^2 + \Pi_2^2) \boldsymbol{v}_i . \tag{57}$$

The term proportional to $d\boldsymbol{v}_i/dt$ is unimportant; it contributes only a negligible correction to the dispersion relation. Wave damping arises from the term proportional to $\boldsymbol{v}_i$. The damping time is of order the time between successive collisions of an ion with neutrals multiplied by the large ratio $v_i/v_{\rm slip}$. Thus,

$$\omega_k t_{\rm damp} \sim \left(\frac{n_i}{n_0}\right)\left(\frac{V_A}{v_{\rm th}}\right)\left(\frac{k_z}{k_\perp}\right)\frac{\Pi_1}{(\Pi_1^2 + \Pi_2^2)} . \tag{58}$$

Making use of equation (56), we may recast equation (58) in the more revealing form

$$\omega_k t_{\rm damp} \sim \left(\frac{n_i}{n_0}\right)\left(\frac{V_A}{v_{\rm th}}\right)\left(\frac{l_0}{L}\right)^{1/3}(l_0 k_\perp)^{-4/3} . \tag{59}$$

2. $\Pi_1 \gg 1$.—The neutrals found at a particular spatial location last collided with ions at locations more distinct than the "perpendicular wavelength" of the shear Alfvén wave under consideration. Therefore, $\boldsymbol{v}_0 = 0$, and the damping time is of order the time between successive collisions of an ion with neutrals. Hence,

$$\omega_k t_{\rm damp} \sim \left(\frac{n_i}{n_0}\right)\left(\frac{V_A}{v_{\rm th}}\right)\left(\frac{k_z}{k_\perp}\right)\Pi_1 . \tag{60}$$

We may rewrite equation (60) in the more revealing form

$$\omega_k t_{\rm damp} \sim \left(\frac{n_i}{n_0}\right)\left(\frac{V_A}{v_{\rm th}}\right)\left(\frac{l_0}{L}\right)^{1/3}(l_0 k_\perp)^{2/3} . \tag{61}$$

As is evident from equations (59) and (61), the critically balanced cascade is most vulnerable to damping by ion-neutral collisions at $k_\perp \sim l_0^{-1}$. Its development to higher perpendicular wave number, $k_\perp \gtrsim l_0^{-1}$, is only possible in environments for which the neutral fraction, $f \equiv n_0/n_i$, is sufficiently small:

$$f_{\rm crit} \sim \frac{V_A}{v_{\rm th}}\left(\frac{l_0}{L}\right)^{1/3} . \tag{62}$$

Taking $\sigma_0 \sim 10^{-15}$ cm^2 yields

$$l_0 \sim 10^{15}\left(\frac{\rm cm^{-3}}{n_i}\right) {\rm cm} . \tag{63}$$

Then assuming $V_A \sim v_{\rm th}$, we arrive at

$$f_{\rm crit} \sim 7 \times 10^{-2}\left(\frac{L}{\rm pc}\right)^{-1/3}\left(\frac{n_i}{\rm cm^{-3}}\right)^{-1/3} . \tag{64}$$

The numerical estimate of $f_{\rm crit}$ implies that the interstellar electron density fluctuations responsible for pulsar scintillations arise in highly ionized components of the interstellar medium.

5.2. *Viscous Damping*

Shear Alfvén waves are resistant to damping in fully ionized plasmas. To linear order, and in the limit that the particles have gyroradii much smaller than the wavelength, they are immune to collisionless damping. They do suffer collisional damping, but only that associated with the small cross field components of the viscosity tensor. The dominant components of the shear tensor on scales in the inertial range of the critically balanced cascade have magnitudes

$$s_\lambda \sim k_\perp v_\lambda \sim k_\perp^{2/3} L^{-1/3} V_A , \tag{65}$$

where we assume $v_L \sim V_A$. Collisions between protons result in cross field components of the kinematic viscosity tensor being of order

$$\gamma \sim \frac{e^2 c^2 n_i}{v_{\rm th} B_0^2} \ln \mathscr{L} , \tag{66}$$

with $\ln \mathscr{L} \sim 20$ denoting the Coulomb logarithm. In a collision-dominated fluid, viscosity cuts off the Kolmogorov inertial range at an inner (viscous) scale, l_v, of order

$$l_v \sim \frac{L}{{\rm Re}^{3/4}} , \tag{67}$$

where the Reynolds number

$${\rm Re} \sim \frac{v_L L}{\gamma} . \tag{68}$$

A numerical comparison of the inner scale so defined with the proton gyroradius,

$$r_i \approx \frac{m_i c v_{\rm th}}{e B_0}, \tag{69}$$

yields

$$\frac{l_v}{r_i} \sim 10^{-1}\left(\frac{n_i}{\rm cm^{-3}}\right)^{9/8}\left(\frac{L}{\rm pc}\right)^{1/4}\left(\frac{B}{10^{-6}\ \rm G}\right)^{-5/4}\left(\frac{T}{10^4\ \rm K}\right)^{-7/8}. \tag{70}$$

This ratio is small compared to unity for almost all conceivable interstellar environments. Thus, as far as we can see, the cascade might reach values of $k_\perp^{-1}$ approaching the proton gyroradius.

5.3. *Generation of Other Modes*

At an early stage in this paper, we discard the pseudo-Alfvén wave by projecting the perturbed velocity and magnetic fields (eq. [14]) onto the polarization vector of the shear Alfvén wave. A similar restriction is made in Paper I. There it has the motivation that, in high β plasmas, the slow magnetosonic wave (of which the pseudo-Alfvén wave is the incompressible limit) undergoes heavy collisionless damping by resonant particles (Barnes 1966). Relying on Barnes' damping to eliminate compressible MHD modes is plausible when dealing with weak MHD turbulence, because the nonlinear interactions act over many wave periods. However, it is not clear that Barnes' damping works fast enough to compete with the nonlinear interactions in cases of strong MHD turbulence. Thus our restriction to shear Alfvén waves is no more than a guess. This guess seems appropriate for solar wind turbulence, but even in this relatively well studied environment the dominance of shear Alfvén waves lacks a convincing explanation. Here we settle for answering a simpler question. Does the generation of pseudo-Alfvén waves divert a significant fraction of the energy from a symmetric, critically balanced cascade of shear Alfvén waves? As shown below, the answer is a resounding no.

Let "S_k" and "P_k" denote shear and pseudo-Alfvén waves with wavevector $\boldsymbol{k}$, respectively. Consider the process $S_k + S_2 \rightarrow S_1$. The rate at which energy is lost from the wave with wavevector $\boldsymbol{k}$ can be read off equation (29). By analogy, there is a similar expression for the rate at which a shear Alfvén wave with wavevector $\boldsymbol{k}$ loses energy by the process

$$S_k + S_2 \rightarrow P_1 . \tag{71}$$

A detailed derivation, which we do not give here, confirms that

$$\partial_t E_k = -\Gamma_k E_k , \tag{72}$$

where

$$\Gamma_k = 4 \int d^3k_1\, d^3k_2\, \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k})(\hat{\boldsymbol{f}}_1 \cdot \hat{\boldsymbol{e}}_k)^2 (\boldsymbol{k} \cdot \hat{\boldsymbol{e}}_2)^2 \mathscr{T}_2 E_2 . \tag{73}$$

The only difference between the above expression and the $E_2 E_k$ term in equation (29) is that $(\hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_k)^2$ has been replaced by $(\hat{\boldsymbol{f}}_1 \cdot \hat{\boldsymbol{e}}_k)^2$ where

$$\hat{\boldsymbol{f}}_k \equiv (\hat{\boldsymbol{e}}_k \times \hat{\boldsymbol{k}}) . \tag{74}$$

is the unit polarization vector for the pseudo-Alfvén wave. This difference is crucial, as we are about to show.

For shear Alfvén waves undergoing a critically balanced cascade, each of the compensating interactions $S_k + S_2 \rightarrow S_1$ and $S_k + S_2 \leftarrow S_1$ acts to modify the energy in mode $\boldsymbol{k}$ at a rate $\sim \omega_k$. The rate at which the generation of pseudo Alfvén waves drains energy from the critically balanced cascade (eq. [71]) is estimated to be

$$\Gamma_k \sim \omega_k \left(\frac{\hat{\boldsymbol{f}}_1 \cdot \hat{\boldsymbol{e}}_k}{\hat{\boldsymbol{e}}_1 \cdot \hat{\boldsymbol{e}}_k}\right)^2 = \left(\frac{k_{1z}}{k_1}\right)^2 \tan^2 \theta_1 . \tag{75}$$

We ignore the angular factor which is of order unity, and use the *locality* of interactions in $\boldsymbol{k}$-space to set $(k_{1z}/k_1)^2 \sim (k_z/k)^2$. In the inertial-range, $k_z \sim k_\perp^{2/3} L^{-1/3} \ll k_\perp$, so

$$\Gamma_k \sim \omega_k \left(\frac{k_z}{k_\perp}\right)^2 \sim \frac{\omega_k}{(k_\perp L)^{2/3}} . \tag{76}$$

Deep in the inertial-range, $k_\perp L \gg 1$, so the critically balanced cascade incurs only a negligible energy loss through the generation of pseudo-Alfvén waves.

We are really interested in compressible MHD which supports, in addition to the incompressible shear Alfvén wave, two compressible modes, the slow wave (the generalization of the pseudo-Alfvén mode) and the fast wave. The argument just outlined, for ignoring production of pseudo-Alfvén waves by the critically balanced cascade, rests on the polarization properties of the modes; for $k_z/k_\perp \ll 1$ and $k'_z/k'_\perp \ll 1$, $|\hat{\boldsymbol{f}}_{k'} \cdot \hat{\boldsymbol{e}}_k| \ll 1$. We expect this result to generalize to cover the production of slow waves in compressible media. That leaves only the generation of fast waves to worry about. However, a simple argument suggests that the critically balanced cascade is also immune to this potential problem. Comparing the dispersion relation for the fast wave, $\omega^2 = \{V_{\rm A}^2 + c_s^2$

$+ [(V_A^2 - c_s^2)^2 + 4V_A^2 c_s^2 \sin^2\theta]^{1/2}\}k^2/2$, with that for the shear Alfvén wave, $\omega = k_z V_A$, we see that shear Alfvén waves with $k_z^{sA} \ll k_\perp^{sA}$ can only generate fast waves having $k_z^f \sim k_\perp^f \sim k_z^{sA} \ll k_\perp^{sA}$. The restricted phase space for these fast waves limits the energy lost in their generation to the same order as that lost in the generation of slow waves. We tentatively conclude that it is a consistent procedure to ignore the fast and slow waves when studying the critically balanced cascade of shear Alfvén waves.

6. ELECTRON DENSITY FLUCTUATIONS AND SCATTERING OF RADIO WAVES

We propose that shear Alfvénic turbulence might achieve a self-regulated state in which there is rough balance between the linear wave periods and the timescale for nonlinear interactions among the waves. There are some noteworthy features of the inertial-range energy spectrum of the *critically balanced* cascade:

1. Unlike the weak 4-wave cascade, the critically balanced cascade allows for a slow growth of k_z with increasing $k_\perp$. The eddies are elongated along the direction of the mean magnetic field with an aspect ratio $k_\perp/k_z \sim (k_\perp L)^{1/3}$. Taking $k_\perp^{-1}$ to be the diffractive scale $\sim 3 \times 10^8$ cm, and setting the outer scale $L \sim$ pc, yields an aspect ratio $\sim 10^3$.
2. The "one-dimensional" energy spectrum, corresponding to the "three-dimensional" spectrum that has been defined in equation (7), is proportional to $k_\perp^{-5/3}$. The critically balanced cascade produces an "anisotropic Kolmogorov" spectrum.

6.1. *Electron Density Fluctuations*

Our attempt to connect interstellar electron density fluctuations to Alfvénic turbulence is based on the relation between turbulence and density variations in Earth's atmosphere. Atmospheric refractivity fluctuations responsible for stellar scintillation and seeing arise from the turbulent mixing of air parcels having different specific entropy. The entropy acts like a passive contaminant; it does not affect the dynamics. The power spectrum of the fluctuations of a passive scalar assumes the form of the energy spectrum of the turbulence (Lesieur 1990). For incompressible hydrodynamic turbulence, this means the one-dimensional power spectrum is proportional to $k^{-5/3}$. By virtue of its small Mach number, atmospheric turbulence is essentially incompressible. This ensures that the dominant contributions to the density fluctuations are made by entropy fluctuations at constant pressure, rather than by pressure fluctuations associated with the Reynolds stress.

The inertial range of the critically balanced cascade of shear Alfvén waves shares many of the characteristics of the inertial range of atmospheric turbulence. The plasma pressure is constant to first-order in the perturbation strength, because the perturbed magnetic field is orthogonal to the mean field. The velocity perturbations are slow in comparison to the speeds of the MHD waves. Because its electrical conductivity is high, the plasma is tied to magnetic field lines, although it can stream along them. It is reasonable to assume that specific entropy acts as a passive contaminant under mixing by shear Alfvén turbulence, and that its power spectrum assumes the form of the energy spectrum of the turbulence. Then the electron density fluctuations reflect fluctuations of specific entropy at constant plasma pressure. At constant pressure, small variations of specific entropy are directly proportional to those of temperature. They are smoothed by thermal conduction. The cross-field thermal diffusivity is promoted by collisions between protons, and has magnitude similar to that of the cross-field kinematic viscosity. Consequently, it is too small to damp density fluctuations in the inertial range (cf. § 5).

6.2. *Scattering of Radio Waves*

Most work concerning the optics of scintillation assumes that the electron density fluctuations follow an isotropic Kolmogorov law, where isotropy is assumed in the interests of simplicity (e.g., Rickett 1990; Narayan 1992). Our model not only accounts for the slope of the ISM electron density spectrum, but also allows for anisotropy in the shapes of the scatter-broadened images.

The scattering of radio waves by highly elongated, aligned, density fluctuations is analogous to the diffractions of light by a series of parallel, thin slits. The dominant scattering is in directions perpendicular to the mean magnetic field. This is a plausible explanation for the elongated, scatter-broadened images of some distant point sources (see, e.g., Rickett 1990; Wilkinson, Narayan, & Spencer 1994; Frail et al. 1994). Of course, these images are much rounder than individual eddies, since they result from the totality of scattering along the line of sight. Considerable averaging results from passage through even a single outer scale length of the turbulence, since the magnetic field perturbations on this scale are of order unity.

Rather special environments are required to produce images of large aspect ratio. The prime example is the solar wind, where near the Sun the magnetic field lines are drawn out almost radially. Images of 3C 279 close to the solar limb are elongated by 6:1 (Narayan, Anantharamaiah, & Cornwell 1989). However, we cannot claim these observations as unambiguous support, because the electron density fluctuation spectrum in this region is flatter than our theory predicts. We suspect this results from a lack of symmetry in the flux of Alfvén waves propagating toward and away from the Sun; the outward flux is known to exceed the inward flux even at 1 AU. The critically balanced cascade is a steady state solution of the kinetic equations regardless of whether there is or is not symmetry between the fluxes of oppositely directed shear Alfvén waves. However, in the absence of symmetry, it is not the unique solution corresponding to a positive flux of energy in $\boldsymbol{k}$-space.

Future, high resolution, *multifrequency* observations offer the possibility of testing predictions our model makes for the frequency-dependent properties of the scatter-broadened images.

7. DISCUSSION

Paper I presents a detailed analysis of weak shear Alfvénic turbulence in an incompressible magnetized fluid. It describes a resonant 4-wave cascade which possesses a couple of remarkable properties. It does not transfer energy along k_z, and the nonlinear interactions strengthen with increasing $k_\perp$, thereby limiting the inertial range. The current paper addresses two obvious, related questions. What happens to a weak cascade when the interaction strengths become of order unity? What is the nature of the turbulence if the initial excitation is strong?

The conjecture of a *critically balanced* shear Alfvénic cascade answers both questions. The turbulence self-regulates so that there is an approximate balance between the (linear) wave period and the (nonlinear) eddy turnover time. By requiring a constant flux of energy in $\boldsymbol{k}$-space, we work out (phenomenologically) the inertial-range energy spectrum of the critically balanced cascade. Two noteworthy features are (1) The spectrum is anisotropic, $k_z \sim k_\perp^{2/3} L^{-1/3}$; (2) The one dimensional energy spectrum is proportional to $k_\perp^{-5/3}$.

Rewriting the equations of incompressible MHD using Elsasser's variables, we derive a kinetic equation describing the interactions of oppositely moving (there is no interaction between waves moving in the same direction) shear Alfvén waves. We specialize to the case where the spectra of oppositely traveling waves is identical; this corresponds to zero cross helicity. The "closure" problem is handled by application of an EDQNM-type approximation. We prove that the critically balanced cascade is a symmetric, stationary solution of the kinetic equation, and that it is associated with a positive flux of energy in $\boldsymbol{k}$-space.

Wave damping due to ion-neutral collisions is most critical for waves whose perpendicular wavelengths are comparable to the mean free path, l_0, of the neutrals. It is only in highly ionized media that the critically balanced cascade escapes being truncated at scales above l_0. As the diffractive scintillation of pulsars is due to electron density fluctuations on scales much smaller than l_0, we conclude that it arises in highly ionized components of the ISM.

We evaluate and dismiss cross field, kinematic viscosity associated with collisions between protons as a significant mechanism for terminating the critically balanced cascade. Unless ion-neutral collisions are significant, we see no reason why the inner scale of shear Alfvén turbulence should be much larger than the ion gyroradius.

The interstellar electron density spectrum is similar to the energy density spectrum of the critically balanced cascade. We argue that this results from the mixing by shear Alfvén waves of plasma having a range of specific entropy. There is a close analogy with the mixing by atmospheric turbulence of air parcels having different specific entropy. Based on this hypothesis, we expect the electron density fluctuations to have long correlation lengths along the direction of the (local) magnetic field. Such structures scatter radio waves anisotropically, and may be responsible for the anisotropic scatter-broadened images observed by VLBI.

In both Paper I and the current manuscript, we ignore compressibility in the technical calculations of turbulent spectra. This is a serious omission when contemplating applications to the ISM. Lack of appropriate calculations does not prevent us from addressing issues related to ISM electron density fluctuations. However, we are stymied by one point. We cannot explain why shear Alfvén waves might be the dominant mode. This is observed to be the case in the interplanetary medium, but is not understood.

Damping of compressive modes may be the answer. Compressive modes are subject to viscous damping in the collisional limit. They also suffer from collisionless damping. Both forms of damping can be severe in plasmas of moderate β. Collisional damping arises from the longitudinal viscosity due to proton-proton scattering (Braginskii 1965). It acts most effectively on waves for which $kl_i \sim 1$, where $l_i \sim 10^{12}(n_i/\text{cm}^{-3})^{-1}$ cm is the proton mean free path. For these most vulnerable waves, the damping rate approaches the wave frequency. The absorption of wave energy by resonant particles causes collisionless damping (Barnes 1966) which affects both fast and slow modes of all wavelengths. Collisionless damping depends sensitively on the angle between $\boldsymbol{k}$ and $\boldsymbol{B}_0$ for each of the two modes. Its rate can approach the wave frequency. Collisional damping might cut off a turbulent cascade of compressive waves near $kl_i \sim 1$. Alternatively, collisionless damping might terminate the cascade at slightly shorter scales. However, it is unclear whether either kind of damping proceeds fast enough to compete with the nonlinear transfer of energy in strong turbulence.

At least in an incompressible medium, the critically balanced cascade of shear Alfvén wave is self-consistent; it does not suffer significant energy drain through the generation of pseudo-Alfvén waves, the only other wave mode. It is plausible, but unproved, that this self-consistency carries over to the compressible case. Compressibility adds yet an additional complication, the steepening of nonlinear Alfvén waves traveling in one direction (Cohen & Kulsrud 1974; Kennel et al. 1988). We are of the opinion that this is not a serious effect, because energy transfer in the crtically balanced cascade takes place on the timescale of a wave period. We expect wave steepening to occur over a much longer timescale, since the energy density in the inertial range is much smaller than the energy density in the ambient magnetic field. Reasonable as this argument may be, it is no substitute for a theory of turbulence appropriate to a compressible medium.

8. COMPARISON TO PREVIOUS WORK

Not surprisingly, plasmas confined by strong magnetic fields in tokamaks and pinches exhibit highly anisotropic turbulence. Montgomery & Turner (1981) assume that a strong mean magnetic field suppresses field-aligned gradients of magnetic and velocity perturbations. They conjecture that the anisotropic turbulence seen in laboratory experiments involves essentially perpendicular field fluctuations; the perpendicular correlation lengths of these fluctuations are assumed to be much shorter than the parallel correlation lengths. Montgomery (1982) argues that this anisotropic state should be described by the Strauss (1976) equations. It must be noted that "critical balance" between parallel and perpendicular timescales is a key assumption in the derivation of the Strauss equations. In this paper we provide a dynamical basis for the critically balanced state. We also derive the inertial range spectrum (eq. [7]), as well as the relationship between parallel and perpendicular correlation lengths (eq. [4]). An alternate approach starting from the Strauss equations should give identical results, although this is not discussed in Montgomery (1982). The limitations of such an approach are that one effectively assumes what is to be derived.

Montgomery & Turner (1981) point out that the 3-wave resonance relations (cf. Paper I) allow only those interactions in which at least one of the 3-waves has zero frequency. They view MHD turbulence as being quasi-two-dimensional with the dynamics controlled by interactions among triplets of zero frequency modes. No estimates of timescales are provided in their paper. We broadly agree with this part of Montgomery & Turner (1981) and also Montgomery (1982); the quasi-two-dimensional shear Alfvén waves of the critically balanced cascade, after suitable nonlinear frequency renormalization, correspond to their zero frequency modes. Montgomery & Turner (1981) go on to hypothesize the existence of an independent, more nearly isotropic cascade of Alfvén waves. The analyses here and in Paper I contradict this conjecture. We find an initially isotropic spectrum of Alfvén waves to evolve naturally into the nearly two-dimensional, critically balanced cascade. Further comparison between our papers and those of

Montgomery & Turner (1981) and Montgomery (1982) is frustrated by the latters lack of detailed calculations, or even a form for the inertial-range spectrum.

Higdon (1984) bases his work on Montgomery (1982). In retrospect, we consider his insight regarding interstellar turbulence as nothing short of prophetic. However, he provides little in the way of relevant dynamics. Higdon argues that the ISM electron density fluctuations are isobaric entropy fluctuations, and that these are mixed as passive contaminants by an underlying turbulent cascade. He points out that, in directions perpendicular to the mean magnetic field, transport coefficients are drastically reduced, thus allowing a turbulent cascade to reach the small scales responsible for diffractive scintillation. Our picture of interstellar turbulence shares many of these features. However, it also advances a detailed model for shear Alfvénic turbulence whose inertial-range energy spectrum matches the power spectrum of the ISM electron density fluctuations.

Montgomery, Brown & Matthaeus (1987) propose that magnetic pressure fluctuations lead directly to a power spectrum of density fluctuations. If $\boldsymbol{b}_1$ is the perturbation of the mean magnetic field, $\boldsymbol{B}_0$, to leading order, the pressure perturbation is $(\boldsymbol{B}_0 \cdot \boldsymbol{b}_1)/4\pi$. Density variations being proportional to pressure variations, one obtains a power spectrum of density fluctuations that has the same form as the power spectrum of the magnetic field fluctuations. No dynamics for the magnetic field fluctuations is given; the one-dimensional spectrum is simply assumed to be proportional to $k^{-5/3}$. For this model to work, $(\boldsymbol{B}_0 \cdot \boldsymbol{b}_1)$ must be nonzero. This is true for the compressive magnetosonic modes but not for shear Alfvén waves. An assessment of the relevance of this mechanism to the ISM must await calculations of the inertial range energy spectra appropriate to the compressive MHD modes. Moreover, it must be demonstrated that turbulent cascades involving compressive waves can survive dissipation.

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