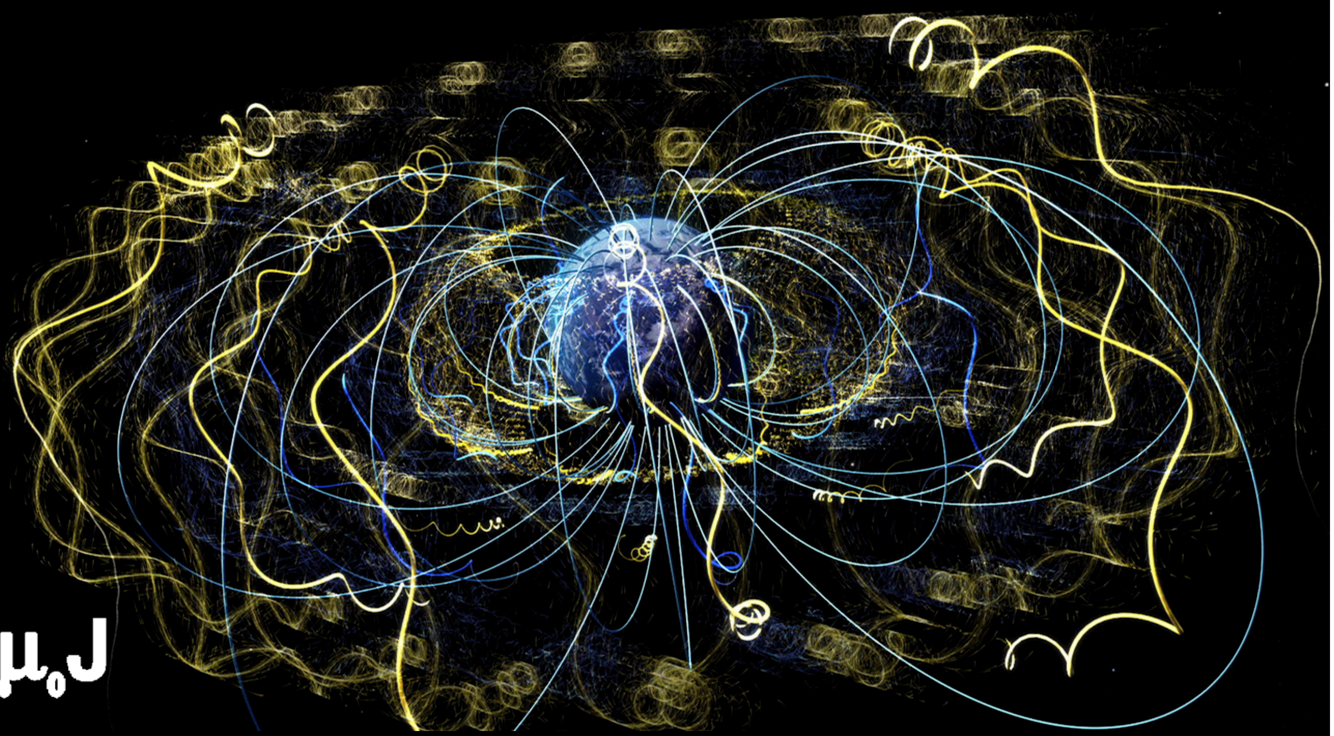
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$



Electricity and Magnetism II: 3812

Professor Jasper Halekas
Van Allen 70
MWF 9:30-10:20 Lecture

9.5 | Guided Waves

- So far we have covered only infinitely extended plane waves

  $\tilde{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ is uniform in plane $\perp$ to $\vec{k}$

- What about localized (confined) waves?

- Consider wave confined in conducting pipe = "wave guide"

- $\vec{E}, \vec{B} = 0$ in perfect conductor

  $\Delta \vec{E}_{\parallel} = 0 \Rightarrow \vec{E}_{\parallel} = 0$ at wall

  $\Delta B_{\perp} = 0 \Rightarrow B_{\perp} = 0$ at wall

  $\sigma_f, \vec{K}_f \neq 0$ at wall

- Seek solutions that propagate along guide

$$\Rightarrow \hat{k} = \hat{z}$$

$$\vec{\tilde{E}}(\vec{r},t) = \vec{\tilde{E}}_0(x,y)\, e^{i(kz-\omega t)}$$
$$\vec{\tilde{B}}(\vec{r},t) = \vec{\tilde{B}}_0(x,y)\, e^{i(kz-\omega t)}$$

- Amplitude a function of x, y

$$\vec{\tilde{E}}_0 = \tilde{E}_{0x}\,\hat{x} + \tilde{E}_{0y}\,\hat{y} + \tilde{E}_{0z}\,\hat{z}$$
$$\vec{\tilde{B}}_0 = \tilde{B}_{0x}\,\hat{x} + \tilde{B}_{0y}\,\hat{y} + \tilde{B}_{0z}\,\hat{z}$$

- Drop ~ for simplicity, but amplitude could be complex

$$\nabla \times \vec{E} = -\partial \vec{B}/\partial t$$

$$\Rightarrow \partial E_{0y}/\partial x - \partial E_{0x}/\partial y = i\omega B_{0z}$$
$$\partial E_{0z}/\partial y - ik E_{0y} = i\omega B_{0x}$$
$$ik E_{0x} - \partial E_{0z}/\partial x = i\omega B_{0y}$$

$$\nabla \times \vec{B} = \frac{1}{c^2}\partial \vec{E}/\partial t$$

$$\Rightarrow \partial B_{0y}/\partial x - \partial B_{0x}/\partial y = \frac{-i\omega}{c^2} E_{0z}$$
$$\partial B_{0z}/\partial y - ik B_{0y} = \frac{-i\omega}{c^2} E_{0x}$$
$$ik B_{0x} - \partial B_{0z}/\partial x = \frac{-i\omega}{c^2} E_{0y}$$

Solve for $B_{0x}, B_{0y}, E_{0x}, E_{0y}$

$$E_{0x} = \frac{i}{(\omega/c)^2 - k^2}\left(k \frac{\partial E_{0z}}{\partial x} + \omega \frac{\partial B_{0z}}{\partial y}\right)$$

$$E_{0y} = \frac{i}{(\omega/c)^2 - k^2}\left(k \frac{\partial E_{0z}}{\partial y} - \omega \frac{\partial B_{0z}}{\partial x}\right)$$

$$B_{0x} = \frac{i}{(\omega/c)^2 - k^2}\left(k \frac{\partial B_{0z}}{\partial x} - \frac{\omega}{c^2} \frac{\partial E_{0z}}{\partial y}\right)$$

$$B_{0y} = \frac{i}{(\omega/c)^2 - k^2}\left(k \frac{\partial B_{0z}}{\partial y} + \frac{\omega}{c^2} \frac{\partial E_{0z}}{\partial x}\right)$$

$$\nabla \cdot \vec{B} = 0, \quad \nabla \cdot \vec{E} = 0$$

$\Rightarrow$

$$\frac{\partial^2 E_{0z}}{\partial x^2} + \frac{\partial^2 E_{0z}}{\partial y^2} + \left((\omega/c)^2 - k^2\right) E_{0z} = 0$$

$$\frac{\partial^2 B_{0z}}{\partial x^2} + \frac{\partial^2 B_{0z}}{\partial y^2} + \left((\omega/c)^2 - k^2\right) B_{0z} = 0$$

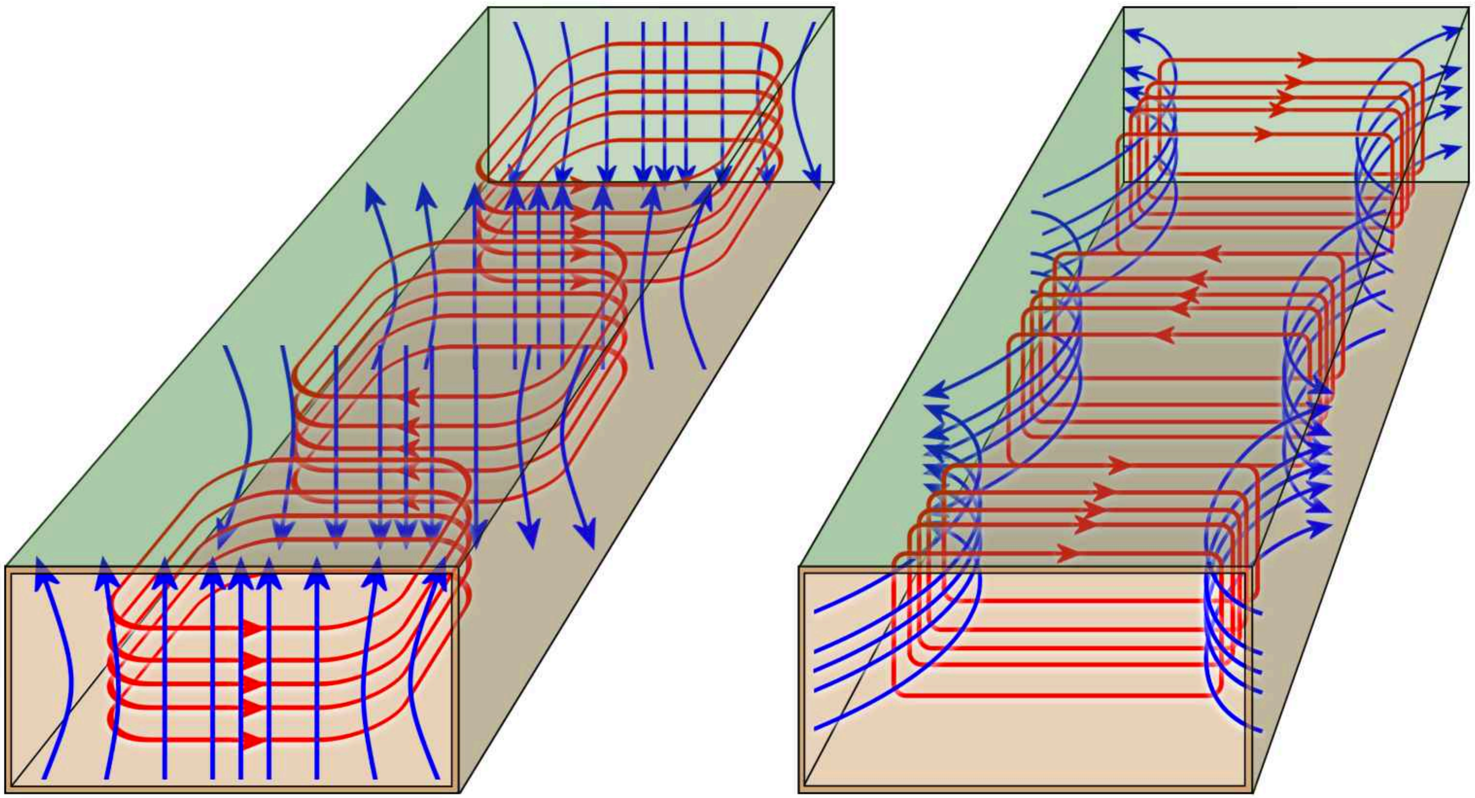
Two special cases:

$E_{0z} = 0 \Rightarrow$ Transverse Electric (TE)

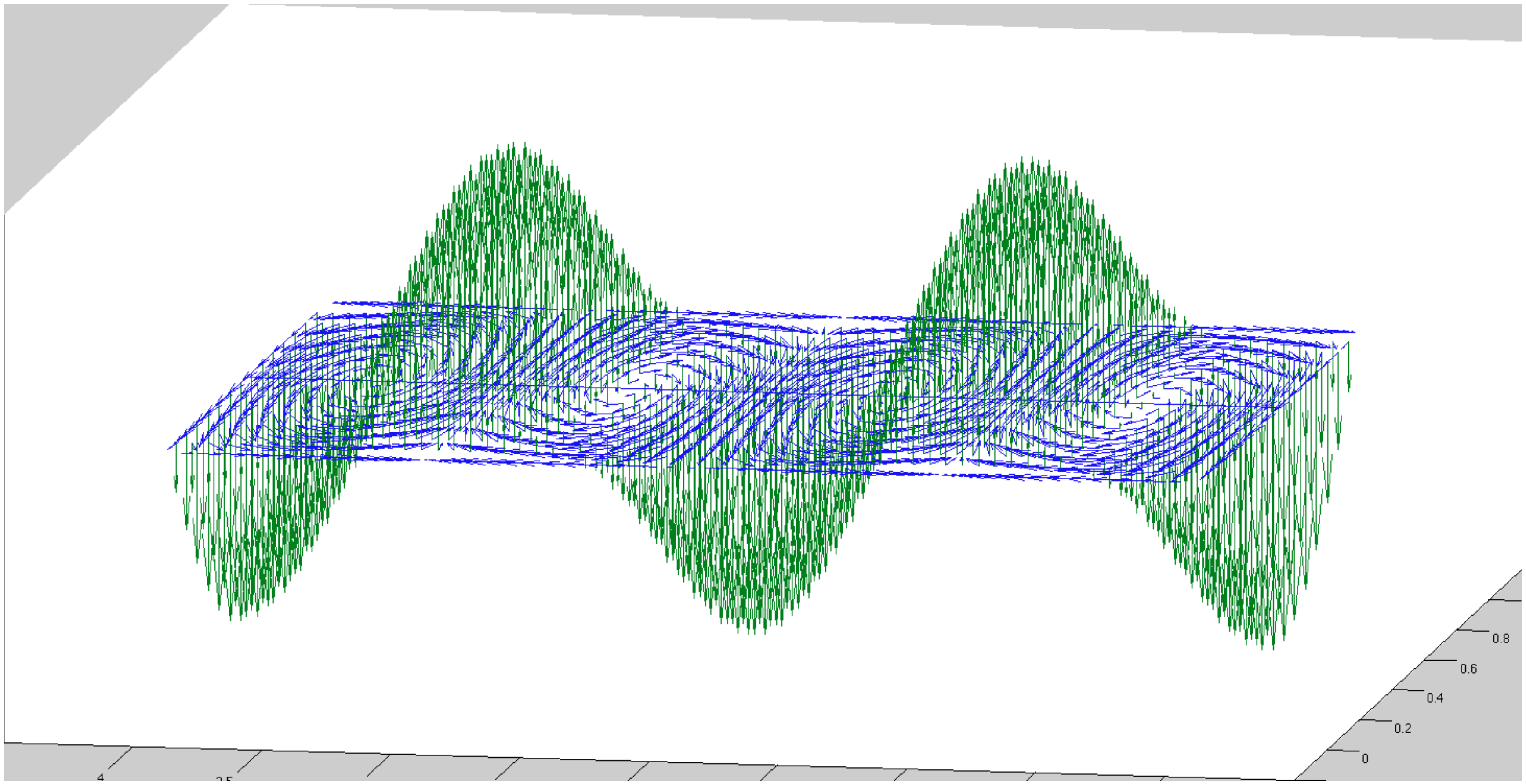
$B_{0z} = 0 \Rightarrow$ Transverse Magnetic (TM)

- Can't have both (TEM) except in a non-hollow guide

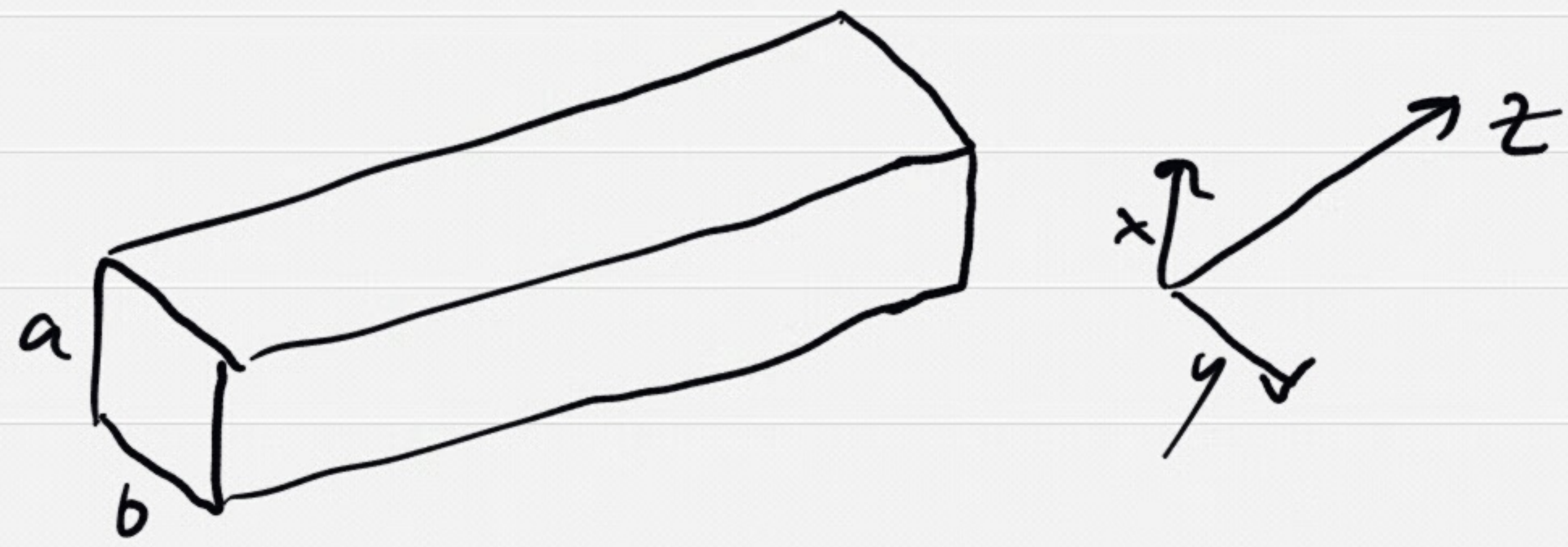
TE/TM Waveguide Solutions



Propagating TE Wave



TE Waves in Rectangular Guide



TE $\Rightarrow E_z = 0$

$$\partial^2 B_{0z}/\partial x^2 + \partial^2 B_{0z}/\partial y^2 + \left(\frac{\omega^2}{c^2} - k^2\right) B_{0z} = 0$$

Separation of Variables

Assume $B_{0z} = X(x)\,Y(y)$

$$\Rightarrow \partial^2 X/\partial x^2\, Y + X\, \partial^2 Y/\partial y^2 + (\omega^2/c^2 - k^2) XY = 0$$

$$\Rightarrow \frac{1}{X}\partial^2 X/\partial x^2 + \frac{1}{Y}\partial^2 Y/\partial y^2 + \omega^2/c^2 - k^2 = 0$$

Must hold for all x, y

$$\Rightarrow \frac{1}{X}\partial^2 X/\partial x^2 + C_1 = 0$$

$$\frac{1}{Y}\partial^2 Y/\partial y^2 + C_2 = 0$$

w/ $C_1 + C_2 = \omega^2/c^2 - k^2$

Write $C_1 = k_x^2$, $C_2 = k_y^2$

$$\Rightarrow \frac{\partial^2 X}{\partial x^2} = -k_x^2 X$$

$$\frac{\partial^2 Y}{\partial y^2} = -k_y^2 Y$$

solutions $X(x) = A\sin(k_x x) + B\cos(k_x x)$

$Y(y) = C\sin(k_y y) + D\sin(k_y y)$

$B_\perp = 0$ at walls

$\Rightarrow B_{0x} = 0$ at $x = a, a$

$B_{0y} = 0$ at $y = 0, b$

and $E_{0z} = 0$ everywhere

These together imply

$\partial B_{0z}/\partial x = 0$ at $x = 0, a$

$\partial B_{0z}/\partial y = 0$ at $y = 0, b$

$$\Rightarrow X(x) = B\cos\left(\frac{m\pi x}{a}\right)$$

$$Y(y) = D\cos\left(\frac{n\pi y}{b}\right)$$

$$B_{0z} = B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

Finally

$$\boxed{\begin{aligned} &B_z(x,y,z,t) \\ &= B_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos(kz-\omega t+\delta) \\ &\quad \text{w/} \quad \frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}=\frac{\omega^2}{c^2}-k^2 \end{aligned}}$$

For $E_z = 0$

$$B_{0x} = \frac{iB_0}{(\omega/c)^2-k^2}\, k\, \partial B_{0z}/\partial x$$

$$= \frac{ikB_0}{(\omega/c)^2-k^2}\cdot -\frac{m\pi}{a}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

$$\Rightarrow B_x(x,y,z,t)$$

$$= \frac{kB_0}{(\omega/c)^2-k^2}\cdot\frac{m\pi}{a}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin(kz-\omega t+\delta)$$

B_x out of phase w/ B_z $\Rightarrow$ $\vec{B}$ forms loops

Cutoff Frequency

$$k^2 = \omega^2/c^2 - \frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2}$$

k imaginary (attenuation)

if $\omega < c\sqrt{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}} = \omega_{mn}$

TE Waveguide Solutions