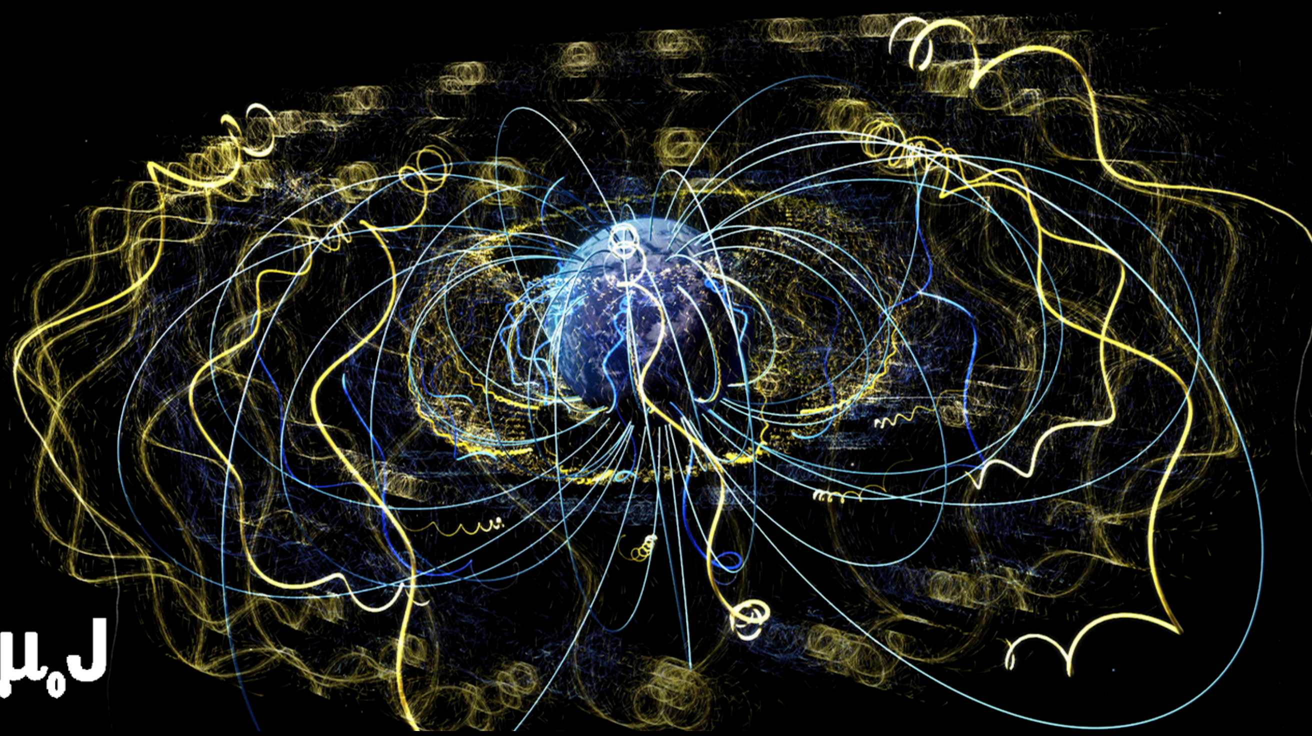
$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

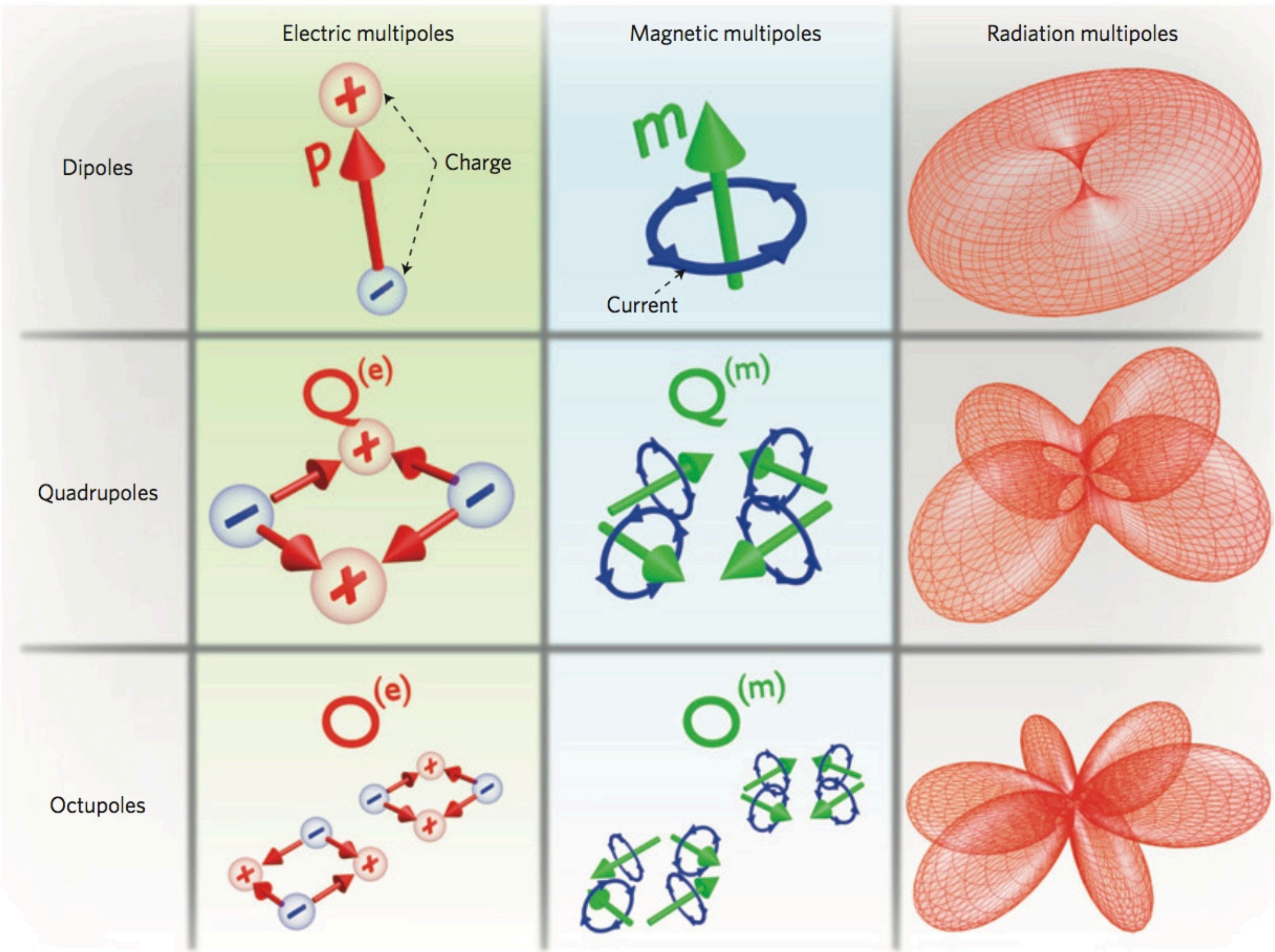
$$\nabla \times B = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} + \mu_0 J$$



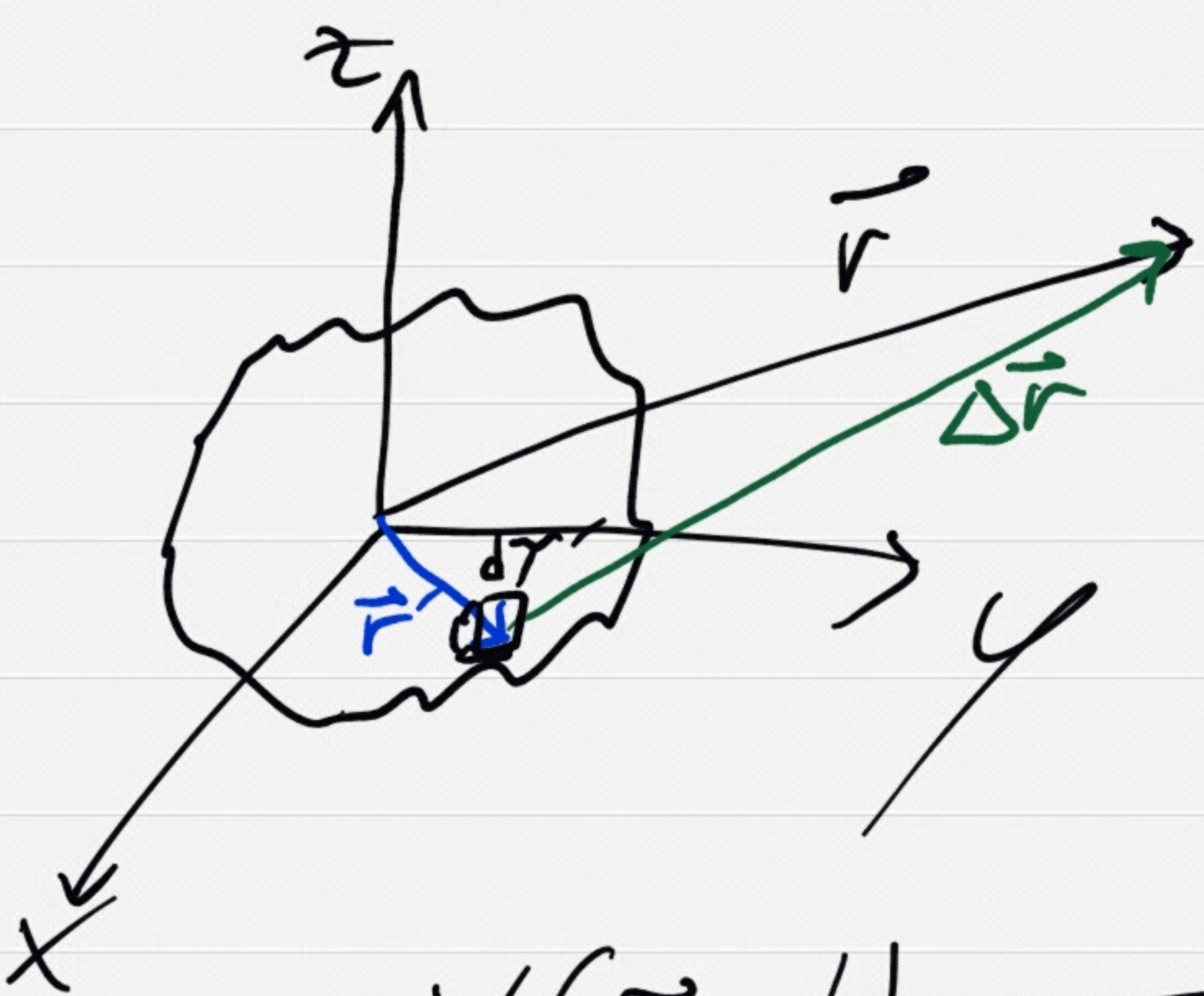
Electricity and Magnetism II: 3812

Professor Jasper Halekas
Virtual by Zoom!
MWF 9:30-10:20 Lecture

Multipole Radiation



Multipole Radiation



$$V(\vec{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,',t_r)}{\Delta r}\,d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{r}\,',t-\Delta r/c)}{\Delta r}\,d\tau'$$

w/ $\Delta r = \sqrt{r^2 + r'^2 - 2\vec{r}\cdot\vec{r}\,'}$

Assume $r' \ll r$ (far from source)

$$\Rightarrow \Delta r \sim r\left(1 - \frac{\vec{r}\cdot\vec{r}\,'}{r^2}\right) = r\left(1 - \frac{\hat{r}\cdot\vec{r}\,'}{r}\right)$$

$$1/\Delta r \sim \frac{1}{r}\left(1 + \frac{\vec{r}\,'\cdot\vec{r}}{r^2}\right) = r\left(1 + \frac{\hat{r}\cdot\vec{r}\,'}{r}\right)$$

$$\rho(\vec{r}\,', t - \Delta r/c) \sim \rho\left(\vec{r}\,', t - \frac{r}{c} + \frac{\hat{r}\cdot\vec{r}\,'}{c}\right)$$

Expand ρ about $t_0 = t - r/c$
(retarded time at origin)

$$\Rightarrow \rho\left(\vec{r}\,', t - \frac{\Delta r}{c}\right) \sim \rho(\vec{r}\,',t_0) + \dot{\rho}(\vec{r}\,',t_0)\cdot\frac{\hat{r}\cdot\vec{r}\,'}{c} + \cdots$$

$$V(\vec{r},t) \sim \frac{1}{4\pi\varepsilon_0} \int \left\{ \frac{1}{r}\left(1 + \frac{\hat{r}\cdot\vec{r}\,'}{r}\right) \cdot \left[\rho(\vec{r}\,',t_0) + \dot{\rho}(\vec{r}\,',t_0)\frac{\hat{r}\cdot\vec{r}\,'}{c} \right] \right\} d\tau'$$

Keep only first order terms in r'

$$\Rightarrow V(\vec{r},t) \sim \frac{1}{4\pi\varepsilon_0 r}\left[\int \rho(\vec{r}\,',t_0)\,d\tau' + \frac{\hat{r}}{r}\cdot\int \vec{r}\,'\rho(\vec{r}\,',t_0)\,d\tau' + \frac{\hat{r}}{c}\cdot\int \vec{r}\,'\dot{\rho}(\vec{r}\,',t_0)\,d\tau' \right]$$

$$\boxed{V(\vec{r},t) \cong \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r} + \frac{\hat{r}\cdot\vec{p}(t_0)}{r^2} + \frac{\hat{r}\cdot\dot{\vec{p}}(t_0)}{rc} \right]}$$

Monopole Dipole radiation

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,',t - \frac{\Delta r}{c})}{\Delta r}\,d\tau'$$

$$\sim \frac{\mu_0}{4\pi r}\int \vec{J}(\vec{r}\,',t_0)\,d\tau'$$

Turns out $\int \vec{J}\,d\tau' = \dot{\vec{p}}$

$$\Rightarrow \boxed{\vec{A}(\vec{r},t) \cong \frac{\mu_0}{4\pi r}\dot{\vec{p}}(t_0)}$$

Radiation Fields

Dropping all $1/r^2$ terms ...

$$\nabla V \cong \nabla \left(\frac{1}{4\pi\varepsilon_0} \frac{\hat{r}\cdot\dot{p}(t_0)}{rc} \right)$$

$$\cong \frac{1}{4\pi\varepsilon_0} \frac{\hat{r}\cdot\ddot{p}(t_0)}{rc} \nabla t_0$$

but $\nabla t_0 = -\frac{1}{c}\nabla r = -\frac{\hat{r}}{c}$

$$\Rightarrow \nabla V \cong -\frac{1}{4\pi\varepsilon_0 c^2} \frac{\hat{r}\cdot\ddot{\vec{p}}(t_0)}{r} \hat{r}$$

$$\nabla \times \vec{A} \cong \frac{\mu_0}{4\pi r} \nabla \times \dot{\vec{p}}(t_0)$$

$$\cong \frac{\mu_0}{4\pi r} \nabla t_0 \times \ddot{\vec{p}}(t_0)$$

$$= -\frac{\mu_0}{4\pi r c} \hat{r} \times \ddot{\vec{p}}(t_0)$$

$$\partial \vec{A}/\partial t \cong \frac{\mu_0}{4\pi} \frac{\ddot{\vec{p}}(t_0)}{r}$$

$$\Longrightarrow \vec{E}(\vec{r},t) \cong \frac{\mu_0}{4\pi r} \left[(\hat{r}\cdot\ddot{\vec{p}})\hat{r} - \ddot{\vec{p}} \right]$$

$$\boxed{\vec{E}(\vec{r},t) \cong \frac{\mu_0}{4\pi r} \left[\hat{r}\times(\hat{r}\times\ddot{\vec{p}}) \right]}$$

$$\boxed{\vec{B}(\vec{r},t) \cong -\frac{\mu_0}{4\pi r c} \hat{r}\times\ddot{\vec{p}}}$$

w/ $\vec{p}$ evaluated at $t_0 = t - r/c$

Spherical Coordinates

$$\vec{E}(r,\theta,t) \cong \frac{\mu_0 \ddot{p}_0(t_0)}{4\pi} \frac{\sin\theta}{r} \hat{\theta}$$

$$\vec{B}(r,\theta,t) \cong \frac{\mu_0 \ddot{p}_0(t_0)}{4\pi c} \frac{\sin\theta}{r} \hat{\varphi}$$

w/ z-axis along $\ddot{\vec{p}}$
(not along $\vec{p}$!)

$$\vec{S}(r,\theta,t) = \frac{1}{\mu_0}(\vec{E} \times \vec{B})$$

$$\boxed{\vec{S}(r,\theta,t) = \frac{\mu_0}{16\pi^2 c}\left[\ddot{p}(t_0)\right]^2 \frac{\sin^2\theta}{r^2}\hat{r}}$$

$$P(r,t) = \oint \vec{S} \cdot d\vec{a}$$

$$\boxed{P(r,t) = \frac{\mu_0}{6\pi c}\left[\ddot{p}(t - r/c)\right]^2}$$

Example 1: Oscillating Dipole

$$p(t) = p_0 \cos(\omega t)$$
$$\ddot{p}(t) = -p_0 \omega^2 \cos(\omega t)$$

$$\Rightarrow P_{rad} = \frac{\mu_0 \omega^4 p_0^2}{6\pi c} \cos^2(\omega(t - r/c))$$

and $\langle P_{rad} \rangle = \frac{\mu_0 \omega^4 p_0^2}{12\pi c}$ same as previous derivation

Example 2: Accelerated Particle

Point charge at position $\vec{d}(t)$

$$\vec{p}(t) = q\vec{d}(t)$$
$$\ddot{\vec{p}}(t) = q\, d^2\vec{d}(t)/dt^2$$
$$= q\vec{a}(t)$$

$$\Rightarrow \boxed{P_{rad} = \frac{\mu_0 q^2 a^2}{6\pi c}}$$ w/ a at retarded time

Larmor Formula!

Check Your Understanding

- Verify Faraday's law $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ for electric dipole radiation:

$$\vec{E} = -\hat{\theta}\frac{p_0}{4\pi\varepsilon_0 r}\frac{\omega^2}{c^2}\sin\theta\cos\omega\left(t - \frac{r}{c}\right)$$

$$\vec{B} = -\frac{\mu_0 p_0 \omega^2}{4\pi r c}\hat{\phi}\sin\theta\cos\omega\left(t - \frac{r}{c}\right)$$

$$\nabla \times \vec{A} = \frac{1}{r\,sin\theta}\left(\frac{\partial(sin\theta\ A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(r\ A_\phi)}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial(r\ A_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\hat{\phi}$$

Check Your Understanding

- Verify Ampere's law $\nabla\times\vec{B} = \mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t}$ for electric dipole radiation:

$$\vec{E} = -\hat{\theta}\frac{p_0}{4\pi\varepsilon_0 r}\frac{\omega^2}{c^2}\sin\theta\cos\omega\left(t-\frac{r}{c}\right)$$

$$\vec{B} = -\frac{\mu_0 p_0 \omega^2}{4\pi r c}\hat{\phi}\sin\theta\cos\omega\left(t-\frac{r}{c}\right)$$

$$\nabla\times\vec{A} = \frac{1}{r\, sin\theta}\left(\frac{\partial(sin\theta\ A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(r\ A_\phi)}{\partial r}\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial(r\ A_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\hat{\phi}$$

Q1: $\nabla \times E = \frac{1}{r} \partial/\partial r (r E_\theta)\, \hat{\varphi}$

$$= \frac{1}{r} \partial/\partial r \left(\frac{-p_0 w^2}{4\pi \varepsilon_0 c^2} \sin\theta \cos(w(t - r/c)) \right) \hat{\varphi}$$

$$= \frac{-p_0 w^3}{4\pi \varepsilon_0 c^3 r} \sin\theta \sin(w(t - r/c))\, \hat{\varphi}$$

$$\partial \vec{B}/\partial t = \frac{\mu_0 p_0 w^3}{4\pi r c} \sin\theta \sin(w(t - r/c))\, \hat{\varphi}$$

$$\frac{1}{\varepsilon_0 c^2} = \mu_0 \quad //$$

Q2: $\nabla \times \vec{B} = \frac{1}{r\sin\theta} \partial/\partial\theta (\sin\theta\, B_\varphi)\, \hat{r}$

$$- \frac{1}{r} \partial/\partial r (r B_\varphi)\, \hat{\theta}$$

- The $\hat{\theta}$ term will match up w/ $\mu_0 \varepsilon_0\, \partial \vec{E}/\partial t$

- But, there's an extra radial part, which would only be compensated by one of the terms we dropped in our derivation

- This term didn't affect $\vec{S}$, as it turns out