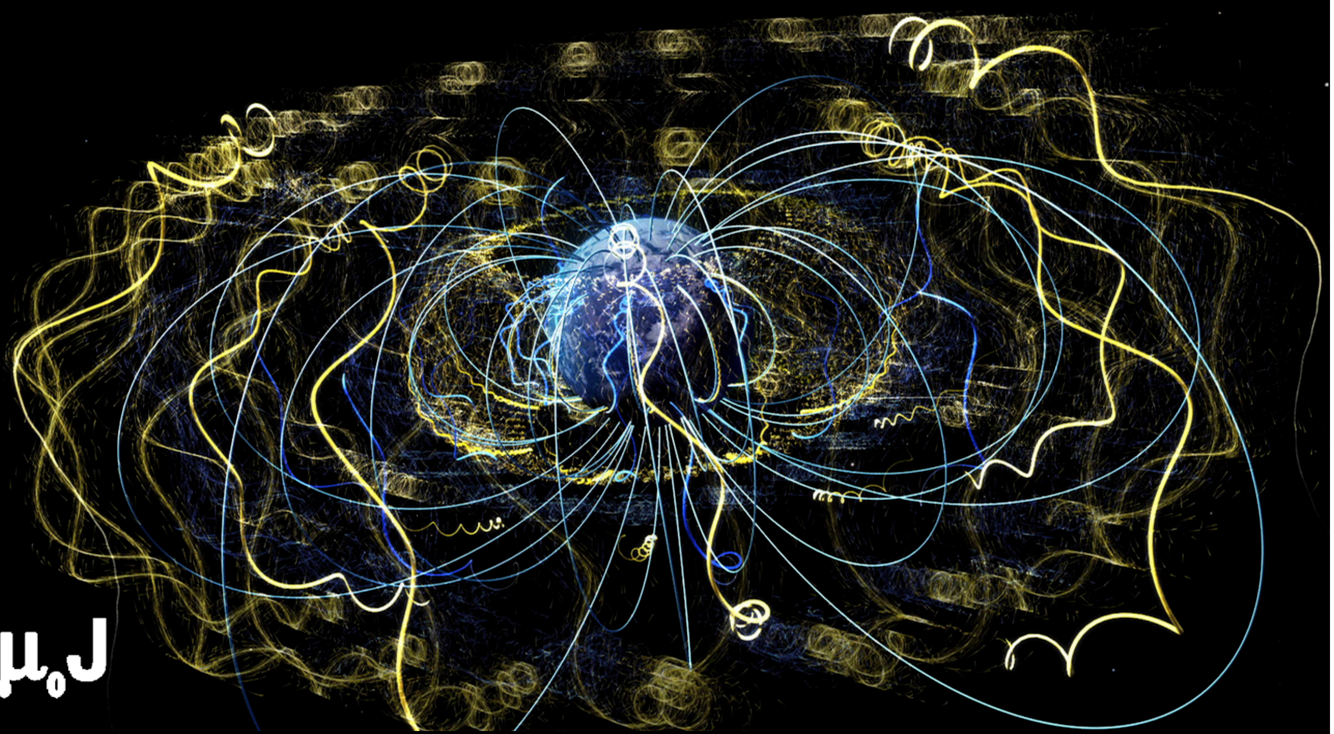
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$



Electricity and Magnetism II: 3812

Professor Jasper Halekas
Virtual by Zoom!
MWF 9:30-10:20 Lecture

Reminders

- Midterm 2 will be Wednesday 4/22
  - The midterm will cover Chapters 9.3-11, except:
    - 9.4.3. The frequency dependence of permittivity
    - 10.1.4. Lorentz force law in potential form
    - 10.2.2. Jefimenko's equations
    - 11.2.2-11.2.3. Radiation reaction
- Equation sheet and sample midterms (with solutions) from last year posted
- Problem solving session last Friday
- Review today

Midterm 2 Rules & Directions

- The exam will be posted on the course web page by 9:30am. You must submit your answers to me by e-mail by 11:30am. The exam is intended to take roughly one hour – the extra hour is grace period to check your work, scan it, and submit it.
- This exam is open book and open notes. However, it is not open internet, and it is not open solutions manual, or open classmate. Please do not utilize solutions, online or otherwise, to solve the problems. I trust you all not to abuse this unique situation.
- Read all the questions carefully and answer every part of each question. Show your work on all problems – partial credit may be granted for correct logic or intermediate steps, even if your final answer is incorrect. Make sure to clearly indicate your final answer.
- Unless otherwise instructed, express your answers in terms of fundamental constants like μ_o and ε_o, rather than calculating numerical values.
- Please ask if you have any questions, including clarification about the instructions, during the exam. The class Zoom meeting will be open during the exam.
- <u>This test is designed to be gender and race neutral.</u>

EM Plane Waves

EM Plane Waves:

Complex : $\vec{E}(\vec{r},t) = \widetilde{E_0} \exp\left(i(\vec{k}\cdot\vec{r} - \omega t)\right)\hat{n}$ $\quad$ $\vec{B}(\vec{r},t) = \frac{k}{\omega}\widetilde{E_0} \exp\left(i(\vec{k}\cdot\vec{r} - \omega t)\right)(\hat{k}\times\hat{n})$

Real: $\vec{E}(\vec{r},t) = E_0 \cos\left(\vec{k}\cdot\vec{r} - \omega t + \delta\right)\hat{n}$ $\quad$ $\vec{B}(\vec{r},t) = \frac{k}{\omega}E_0 \cos\left(\vec{k}\cdot\vec{r} - \omega t + \delta\right)(\hat{k}\times\hat{n})$

$\frac{\omega}{k} = c = \frac{1}{\sqrt{\mu_0\varepsilon_0}}$ $\quad$ $\langle u\rangle = \frac{1}{2}\varepsilon_0 {E_0}^2$ $\quad$ $\langle\vec{S}\rangle = c\langle u\rangle\hat{k} = I\hat{k} = \frac{1}{2}c\varepsilon_0 {E_0}^2\hat{k}$ $\quad$ $\langle\vec{g}\rangle = \frac{\langle\vec{S}\rangle}{c^2} = \frac{1}{2c}\varepsilon_0 {E_0}^2\hat{k}$

<u>Linear materials:</u> $\frac{\omega}{k} = v = \frac{1}{\sqrt{\mu\varepsilon}}$ $\quad$ $\langle\vec{S}\rangle = v\langle u\rangle\hat{k} = I\hat{k} = \frac{1}{2}v\varepsilon {E_0}^2\hat{k}$ $\quad$ $n = \frac{c}{v} = \sqrt{\frac{\mu\varepsilon}{\mu_0\varepsilon_0}}$

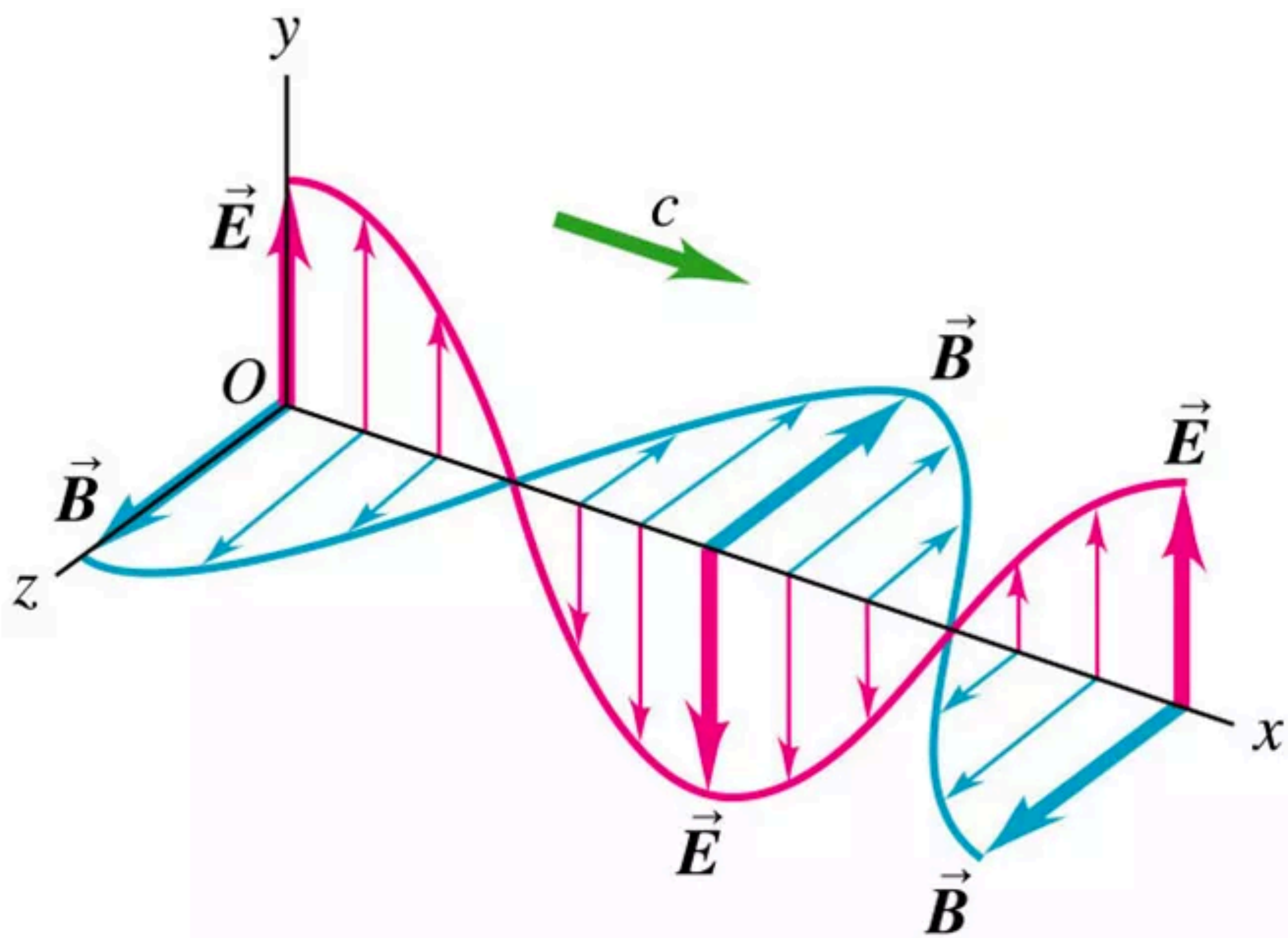
<u>Conductors</u>: $k_r = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1 + (\sigma/\varepsilon\omega)^2} + 1\right]^{1/2}$ $\quad$ $k_i = \omega\sqrt{\frac{\mu\varepsilon}{2}}\left[\sqrt{1 + (\sigma/\varepsilon\omega)^2} - 1\right]^{1/2}$

$\tilde{k} = k_r + ik_i$ $\quad$ $d = 1/k_i$ $\quad$ $n = \frac{c}{v} = \frac{ck_r}{\omega}$ $\quad$ $\frac{B_0}{E_0} = \frac{|k|}{\omega} = \frac{\sqrt{{k_r}^2 + {k_i}^2}}{\omega}$

<u>Reflection/Transmission:</u> $\theta_I = \theta_R$ $\quad$ $\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}$ $\quad$ $\alpha = \frac{\cos\theta_T}{\cos\theta_I}$ $\quad$ $\beta = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{\mu_1 v_1}{\mu_2 v_2}$

p-polarized: $\tilde{E}_{0R} = \frac{\alpha-\beta}{\alpha+\beta}\tilde{E}_{0I}$, $\tilde{E}_{0T} = \frac{2}{\alpha+\beta}\tilde{E}_{0I}$ $\quad$ $R = \langle S_R\rangle/\langle S_I\rangle$ $\quad$ $T = \langle S_T\rangle\cos\theta_T/\langle S_I\rangle\cos\theta_I$

Plane Wave Fields



Complex Notation

Mathematical Representation of Polarized Light

$$\tilde{\mathbf{E}} = E_{0x} e^{i(kz-\omega t+\varphi_x)} \hat{\mathbf{x}} + E_{0y} e^{i(kz-\omega t+\varphi_y)} \hat{\mathbf{y}}$$

$$\tilde{\mathbf{E}} = [E_{0x} e^{i\varphi_x} \hat{\mathbf{x}} + E_{0y} e^{i\varphi_y} \hat{\mathbf{y}}] e^{i(kz-\omega t)}$$

$$\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}$$

$\tilde{\mathbf{E}}_0 = [E_{0x} e^{i\varphi_x} \hat{\mathbf{x}} + E_{0y} e^{i\varphi_y} \hat{\mathbf{y}}]$ = complex amplitude vector for the polarized wave

Linear Dielectrics

Maxwell's Equations in Dielectric medium

$$\text{(i)}\ \nabla \cdot \mathbf{D} = \rho_f, \qquad \text{(iii)}\ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\text{(ii)}\ \nabla \cdot \mathbf{B} = 0, \qquad \text{(iv)}\ \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}.$$

with $D = \varepsilon E$ $\qquad$ $H = \frac{1}{\mu} B$

$\varepsilon = \varepsilon_0 (1 + \chi_e) = \varepsilon_0 \varepsilon_r$ $\qquad$ $\mu \equiv \mu_0 (1 + \chi_m)$

$$\nabla \times \nabla \times \mathbf{E} = -\nabla^2 \mathbf{E} =$$

$$= -\mu \frac{\partial}{\partial t} (\nabla \times \mathbf{H})$$

$$= -\mu \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \right)$$

$$= -\mu\varepsilon \frac{\partial}{\partial t} \left(\frac{\partial \mathbf{E}}{\partial t} \right)$$

$$\Rightarrow \nabla^2 \mathbf{E} = \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Conductors

Electromagnetic Waves in Conductors

According to Ohm's law, the (free) current density in a conductor is proportional to the electric field:

$$\mathbf{J}_f = \sigma \mathbf{E}.$$

plane-wave solutions,

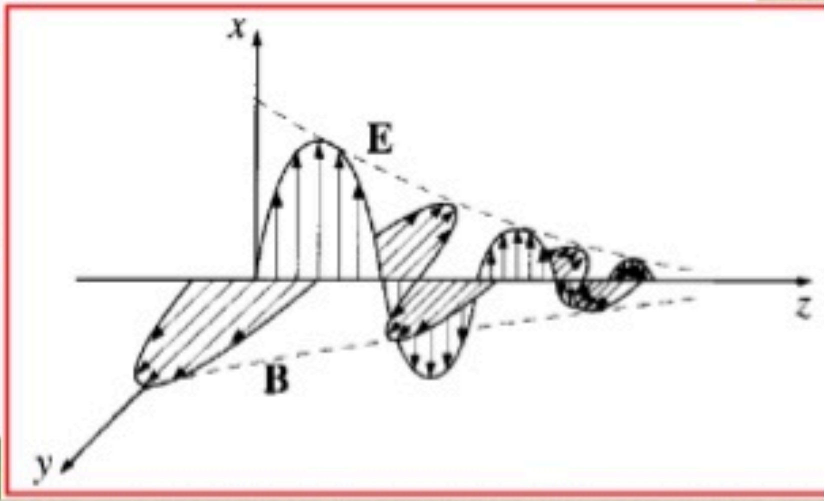
$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(\tilde{k}z-\omega t)},$$

$$\tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(\tilde{k}z-\omega t)},$$

The (real) electric and magnetic fields are, finally,

$$\mathbf{E}(z,t) = E_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E)\,\hat{\mathbf{x}},$$

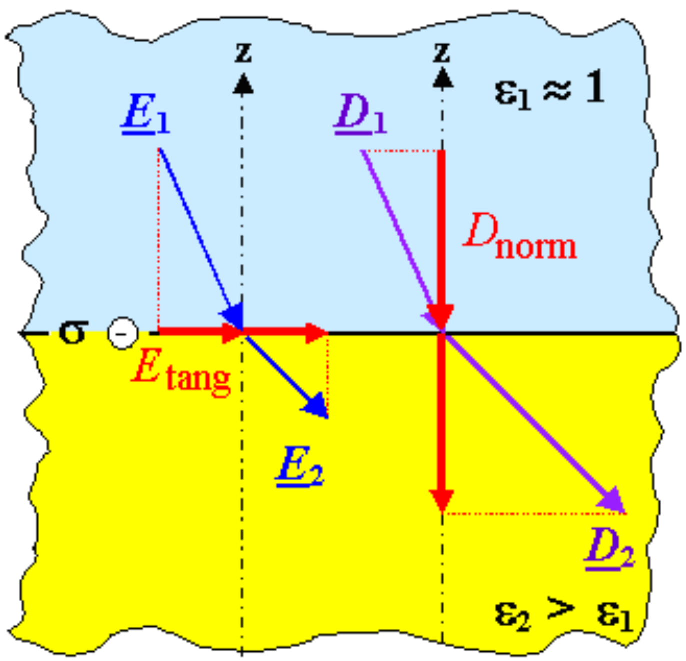
$$\mathbf{B}(z,t) = B_0 e^{-\kappa z}\cos(kz-\omega t+\delta_E+\phi)\,\hat{\mathbf{y}}.$$

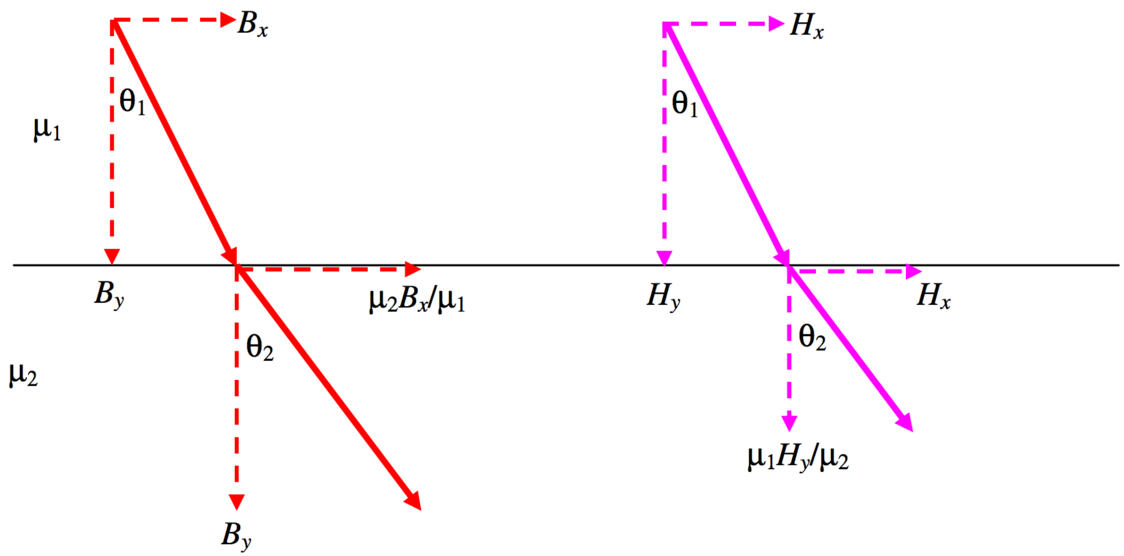


Boundary Conditions-> R,T

Boundary Conditions: $\Delta D_\perp = \sigma_f$ $\quad \Delta \vec{E}_{||} = 0$ $\quad \Delta \vec{D}_{||} = \Delta \vec{P}_{||}$

$\Delta \vec{H}_{||} = \vec{K}_f \times \hat{n}$ $\quad \Delta B_\perp = 0$ $\quad \Delta H_\perp = -\Delta M_\perp$

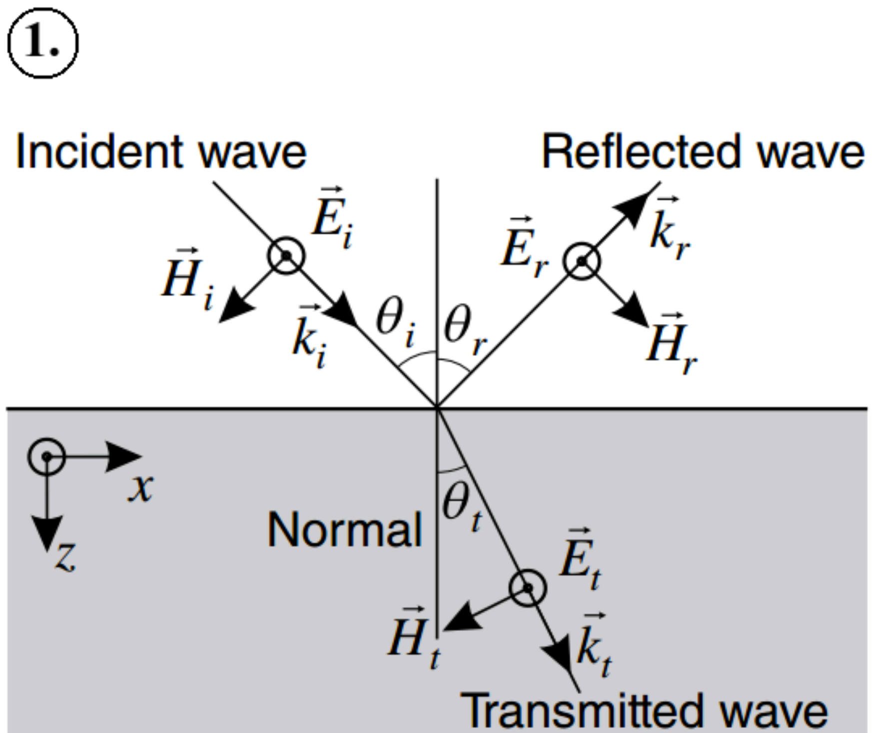


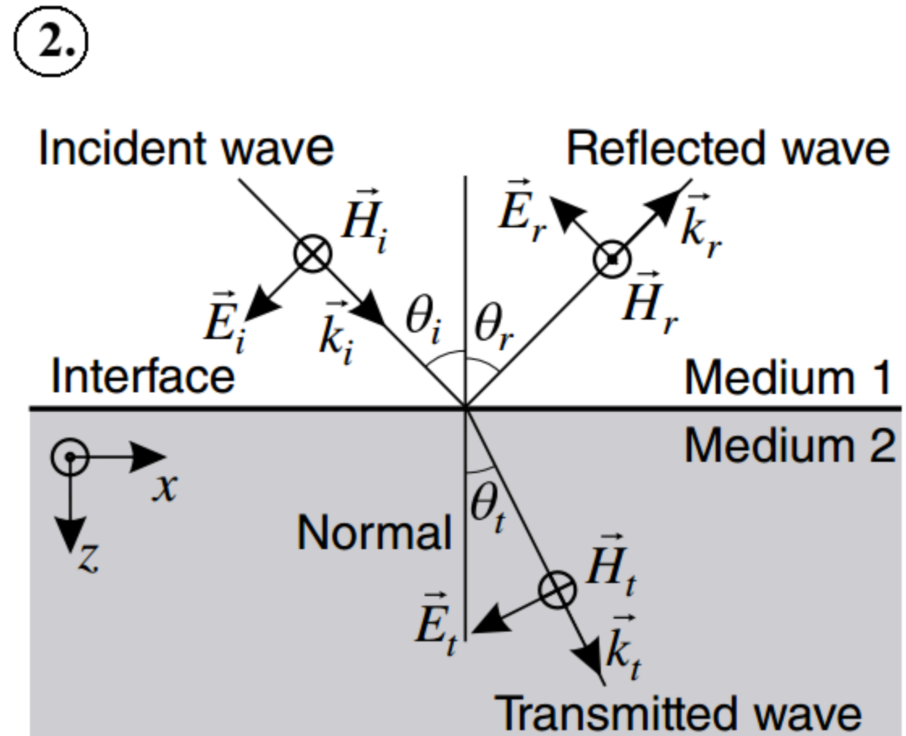


Images show boundary conditions for case w/ no free charge or current at the boundary

Reflection & Transmission at Oblique Incidence

Out-of-Plane (S) Polarization | In-Plane (P) Polarization





Potentials

Potentials: $\vec{B} = \nabla \times \vec{A}$ $\quad \vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$ $\quad$ Gauge Transformation: $\vec{A}' = \vec{A} + \nabla\lambda, V' = V - \frac{\partial\lambda}{\partial t}$

$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = -\frac{\rho}{\varepsilon_0}$ $\qquad (\nabla^2 \vec{A} - \mu_0\varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}) - \nabla(\nabla \cdot \vec{A} + \mu_0\varepsilon_0 \frac{\partial V}{\partial t}) = -\mu_0 \vec{J}$

Coulomb Gauge: $\nabla \cdot \vec{A} = 0$ => $\quad \nabla^2 V = -\frac{\rho}{\varepsilon_0}$ $\qquad (\nabla^2 \vec{A} - \mu_0\varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}) - \mu_0\varepsilon_0 \nabla(\frac{\partial V}{\partial t}) = -\mu_0 \vec{J}$

Lorenz Gauge: $\nabla \cdot \vec{A} = -\mu_0\varepsilon_0 \frac{\partial V}{\partial t}$ => $\quad \nabla^2 V - \mu_0\varepsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$ $\qquad (\nabla^2 \vec{A} - \mu_0\varepsilon_0 \frac{\partial^2 \vec{A}}{\partial t^2}) = -\mu_0 \vec{J}$

Retarded Potentials: $t_r = t - \frac{\Delta r}{c}$ $\quad V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\vec{r}\,', t_r)}{\Delta r} d\tau'$ $\quad \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,', t_r)}{\Delta r} d\tau'$

Liénard-Wiechert Potentials: $V(\vec{r}, t) = \frac{1}{4\pi\varepsilon_0} \frac{qc}{(c\,\Delta r - \overrightarrow{\Delta r} \cdot \vec{v})}$ $\quad \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{qc\vec{v}}{(c\,\Delta r - \overrightarrow{\Delta r} \cdot \vec{v})} = \frac{\vec{v}}{c^2} V(\vec{r}, t)$

$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\varepsilon_0} \frac{\Delta r}{(\overrightarrow{\Delta r} \cdot \vec{u})^3} \left[(c^2 - v^2)\vec{u} + \overrightarrow{\Delta r} \times (\vec{u} \times \vec{a})\right]$ $\qquad \vec{B}(\vec{r}, t) = \frac{1}{c} \widehat{\Delta r} \times \vec{E}$ $\qquad \vec{u} = c\widehat{\Delta r} - \vec{v}$

Constant Velocity: $\quad \vec{E}(\vec{r}, t) = \frac{q}{4\pi\varepsilon_0} \frac{1 - v^2/c^2}{(1 - v^2 \sin^2\theta / c^2)^{3/2}} \frac{\hat{R}}{R^2}$ $\qquad \vec{B}(\vec{r}, t) = \frac{1}{c} \widehat{\Delta r} \times \vec{E} = \frac{1}{c^2} \vec{v} \times \vec{E}$

Scalar and Vector Potentials

$$\left.\begin{matrix}\rho(\bar{r},t)\\ \bar{J}(\bar{r},t)\end{matrix}\right\} \rightarrow \begin{cases}\bar{E}(\bar{r},t)\\ \bar{B}(\bar{r},t)\end{cases}$$

or

$$\searrow \begin{cases}V(\bar{r},t)\\ \bar{A}(\bar{r},t)\end{cases}$$

Maxwell's eqs

$$\nabla\cdot\bar{E}=\frac{\rho}{\varepsilon_0} \qquad \nabla\times\bar{B}=\mu_0\bar{J}+\mu_0\varepsilon_0\frac{\partial\bar{E}}{\partial t}$$

MS $\quad \nabla\cdot\bar{B} \longrightarrow \bar{B}=\nabla\times\bar{A}$

ES $\quad \nabla\times\bar{E}=0 \longrightarrow \bar{E}=-\nabla V$

$$\nabla\cdot\bar{B}=0 \qquad \nabla\times\bar{E}=-\frac{\partial\bar{B}}{\partial t}$$

$$\downarrow \qquad \downarrow$$

$$\boxed{\bar{B}=\nabla\times\bar{A}} \qquad \nabla\times\bar{E}=-\frac{\partial}{\partial t}(\nabla\times\bar{A})$$

$$\nabla\times(\bar{E}+\frac{\partial}{\partial t}\bar{A})=0$$

$$\bar{E}+\frac{\partial}{\partial t}\bar{A}=-\nabla V \longrightarrow \boxed{\bar{E}=-\nabla V-\frac{\partial\bar{A}}{\partial t}}$$

Gauge Transformations

MAGNETIC VECTOR POTENTIAL $\vec{A}$ AND ELECTRIC VECTOR POTENTIAL V

From $\nabla \cdot \vec{B} = 0 \quad \Rightarrow \quad \boxed{\vec{B} = \nabla \times \vec{A} = \nabla \times \left(\vec{A} + \nabla \varphi\right)}$

$\vec{A}$ - ***Magnetic Vector Potential that generates the field. Not unique since*** $\vec{A} + \nabla \varphi$ ***gives the same result (where*** φ ***is any continuous scalar)***

Also

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \Downarrow$$

$$0 = \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} \quad = \quad \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = \nabla \times \left(\vec{E} + \frac{\partial\left(\vec{A} + \nabla \varphi\right)}{\partial t}\right)$$

From this equation we can derive the Scalar Potential V such that

$$\boxed{\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \nabla V = -\frac{\partial}{\partial t}\left(\vec{A} + \nabla \varphi\right) - \nabla\left(V - \frac{\partial \varphi}{\partial t}\right)}$$

The Electromagnetic Field is fully defined by the Vector Potential $\vec{A}$ ***and the Scalar Potential V (or by*** $V - \frac{\partial \varphi}{\partial t}$ ***and*** $\vec{A} + \nabla \varphi$ ***- called a gauge transformation).***

Lienard-Wiechert Potentials

$$V(\mathbf{r},t)=\frac{1}{4\pi\varepsilon_0}\frac{qc}{|\mathbf{r}-\mathbf{w}(t_r)|c-(\mathbf{r}-\mathbf{w}(t_r))\cdot\mathbf{v}(t_r)}$$

implicit equation for *retarded time* t_r

$$c(t-t_r)=|\mathbf{r}-\mathbf{w}(t_r)|$$

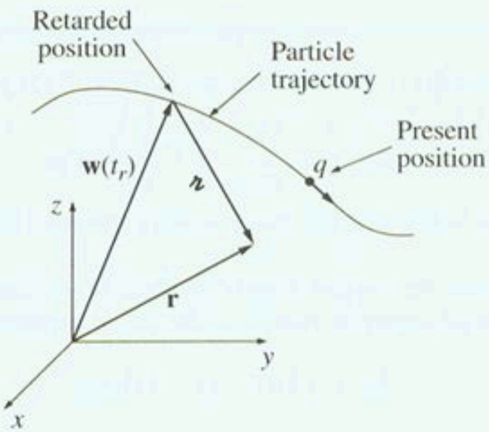
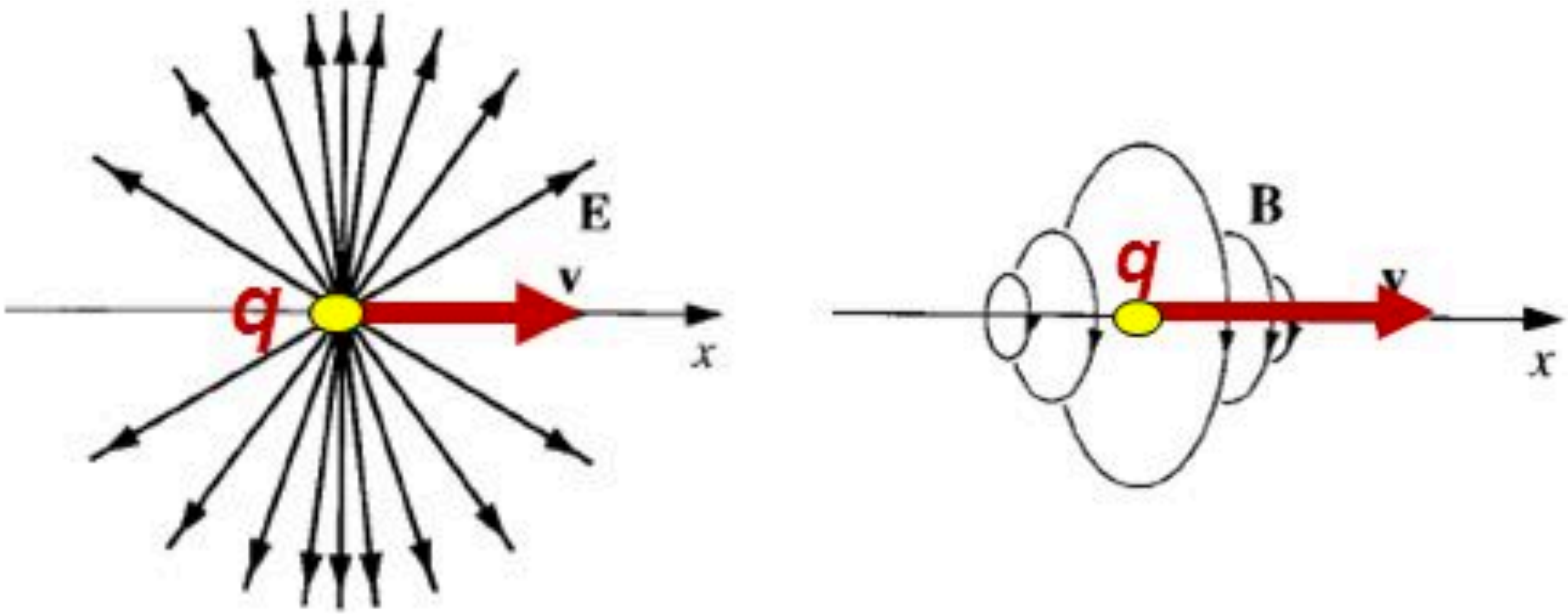


Figure 10.6

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu_0}{4\pi}\frac{qc\mathbf{v}(t_r)}{|\mathbf{r}-\mathbf{w}(t_r)|c-(\mathbf{r}-\mathbf{w}(t_r))\cdot\mathbf{v}(t_r)}$$

$$=\frac{\mathbf{v}(t_r)}{c^2}V(\mathbf{r},t)$$

E & B of Uniformly Moving Point Charge



Radiation

Radiation: Electric Dipole: $\vec{S}(\vec{r},t) \cong \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\frac{sin\theta}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}^2\hat{r}$ $P = \frac{\mu_0 {p_0}^2\omega^4}{12\pi c}$

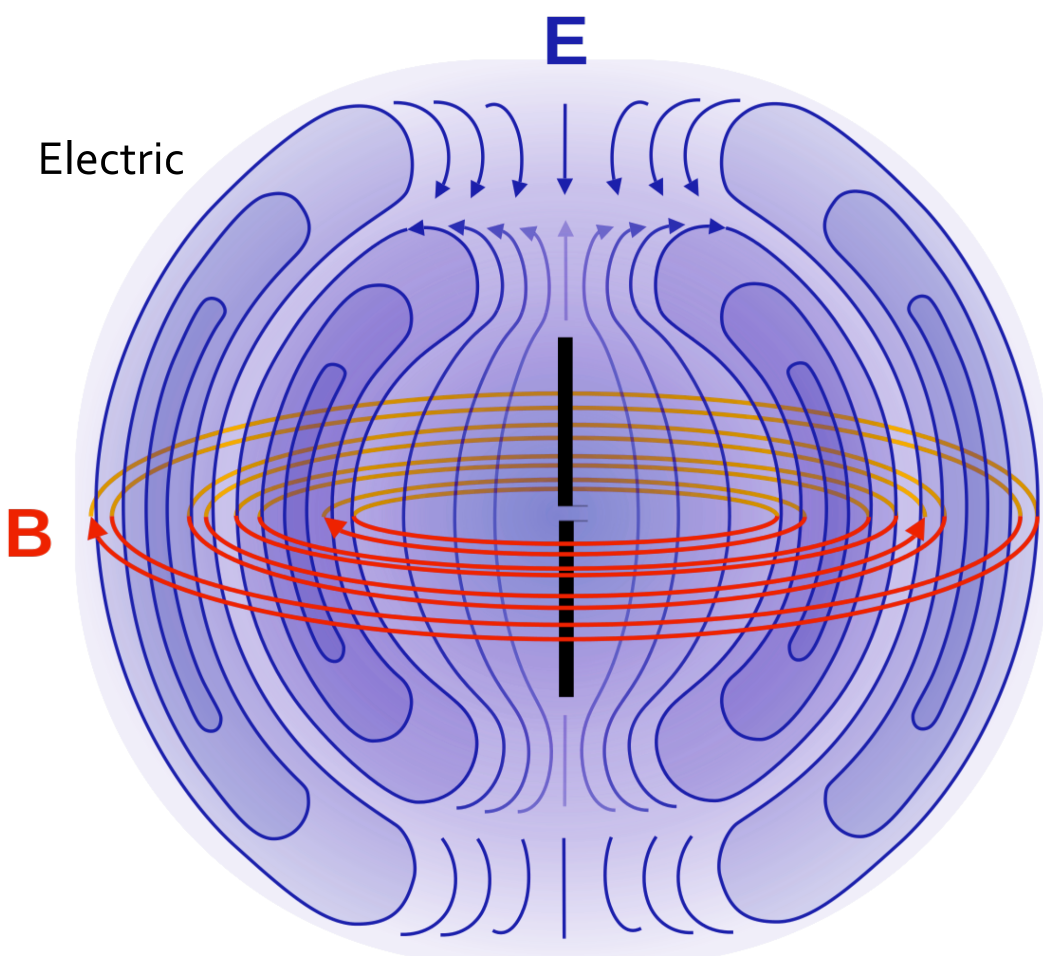
Magnetic Dipole: $\vec{S}(\vec{r},t) \cong \frac{\mu_0}{c}\left\{\frac{m_0\omega^2}{4\pi c}\frac{sin\theta}{r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}^2\hat{r}$ $P = \frac{\mu_0 {m_0}^2\omega^4}{12\pi c^3}$

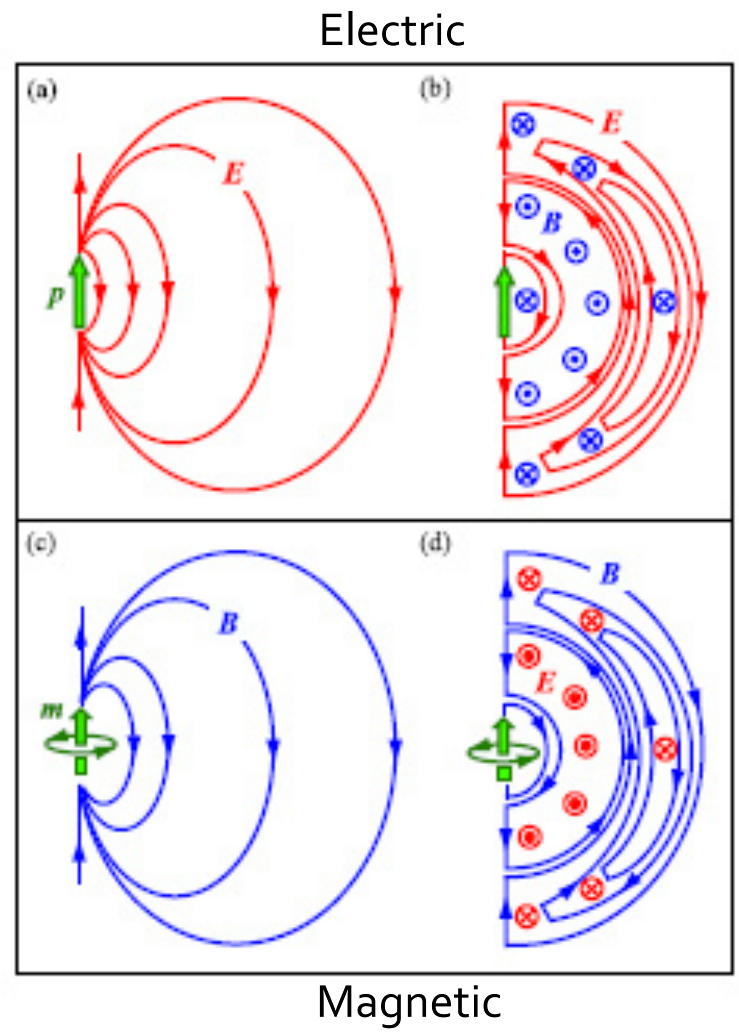
Multipole: $\vec{E}(\vec{r},t) \cong \frac{\mu_0}{4\pi r}[\hat{r}\times(\hat{r}\times\ddot{\vec{p}})]$ $\vec{B}(\vec{r},t) \cong -\frac{\mu_0}{4\pi rc}[\hat{r}\times\ddot{\vec{p}}]$ $\vec{S}(\vec{r},t) \cong \frac{\mu_0\ddot{p}^2}{16\pi^2 c}\frac{sin^2\theta}{r^2}\hat{r}$

Pt. Chrg: $\vec{E}_{rad}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{\Delta r}{(\overrightarrow{\Delta r}\cdot\vec{u})^3}\left[\overrightarrow{\Delta r}\times(\vec{u}\times\vec{a})\right]$ $\vec{S}_{rad} = \frac{1}{\mu_0 c}{E_{rad}}^2\widehat{\Delta r}$ $\mathrm{P} = \frac{\mu_0 q^2\gamma^6}{6\pi c}\left(a^2 - \left|\frac{\vec{v}\times\vec{a}}{c}\right|^2\right)$

Low Velocity: $\vec{S}_{rad} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{sin^2\theta}{\Delta r^2}\widehat{\Delta r}$ $P = \oint\vec{S}\cdot\overrightarrow{da} = \frac{\mu_0 q^2 a^2}{6\pi c}$

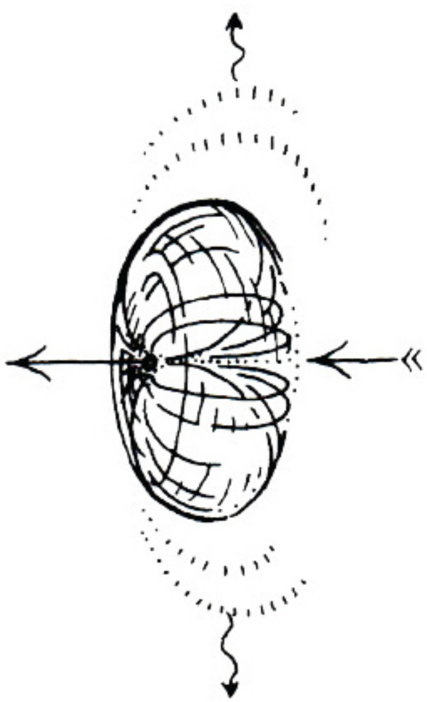
Dipole Radiation





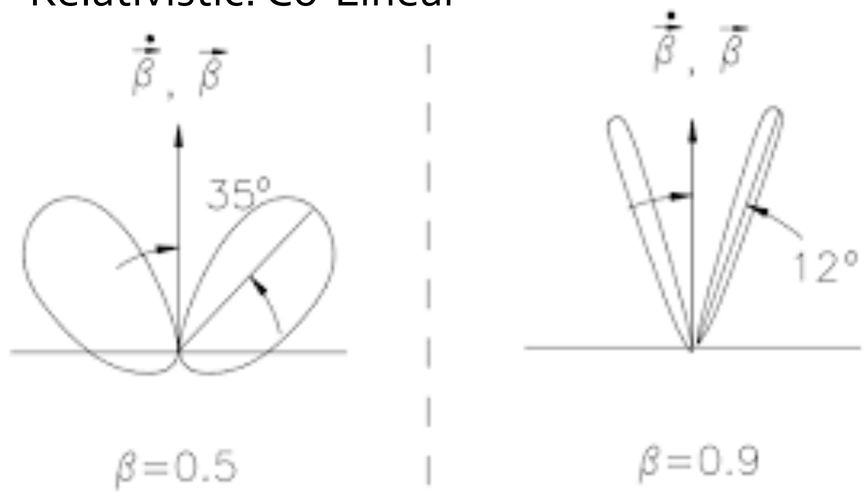
Accelerated Point Charge

RADIATION FROM A POINT-CHARGE



AN ACCELERATING POINT-CHARGE RELEASES A FRONT OF RADIATION IN THE SHAPE OF A TOROID.

Relativistic: Co-Linear



Relativistic: Perpendicular