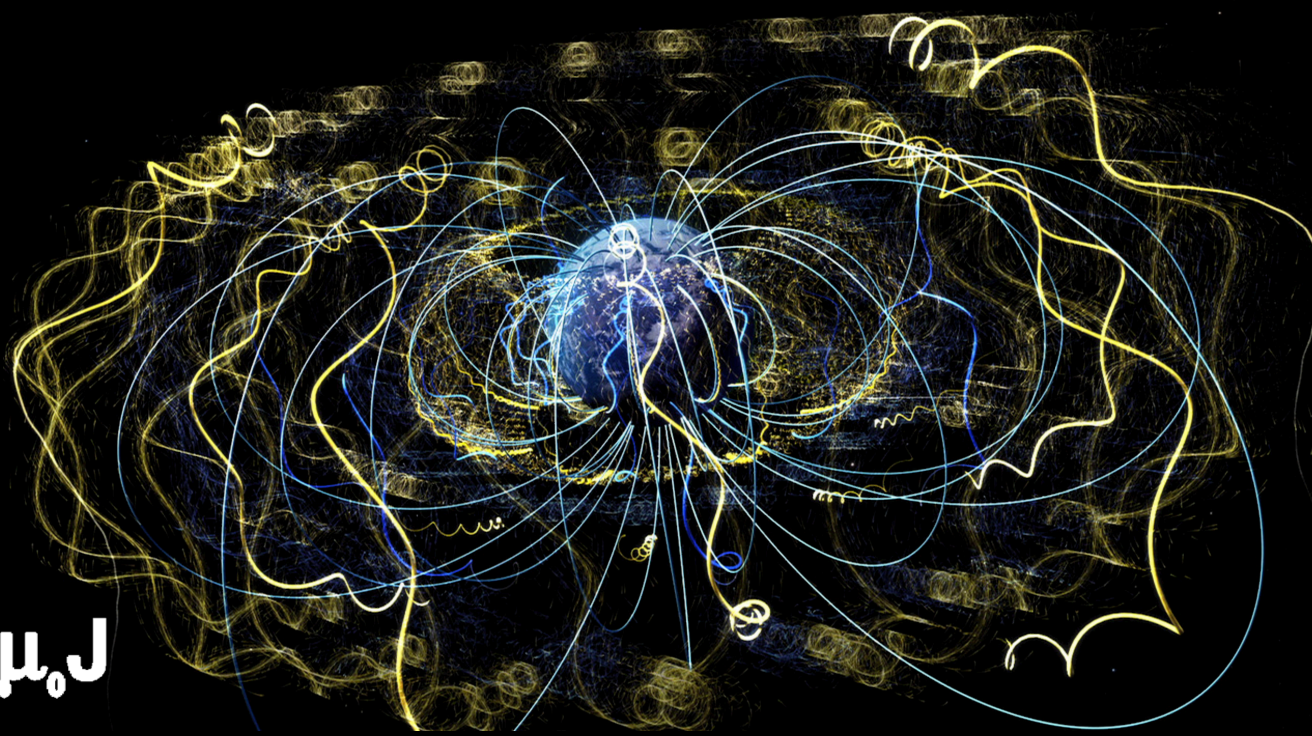
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{J}$$



Electricity and Magnetism II: 3812

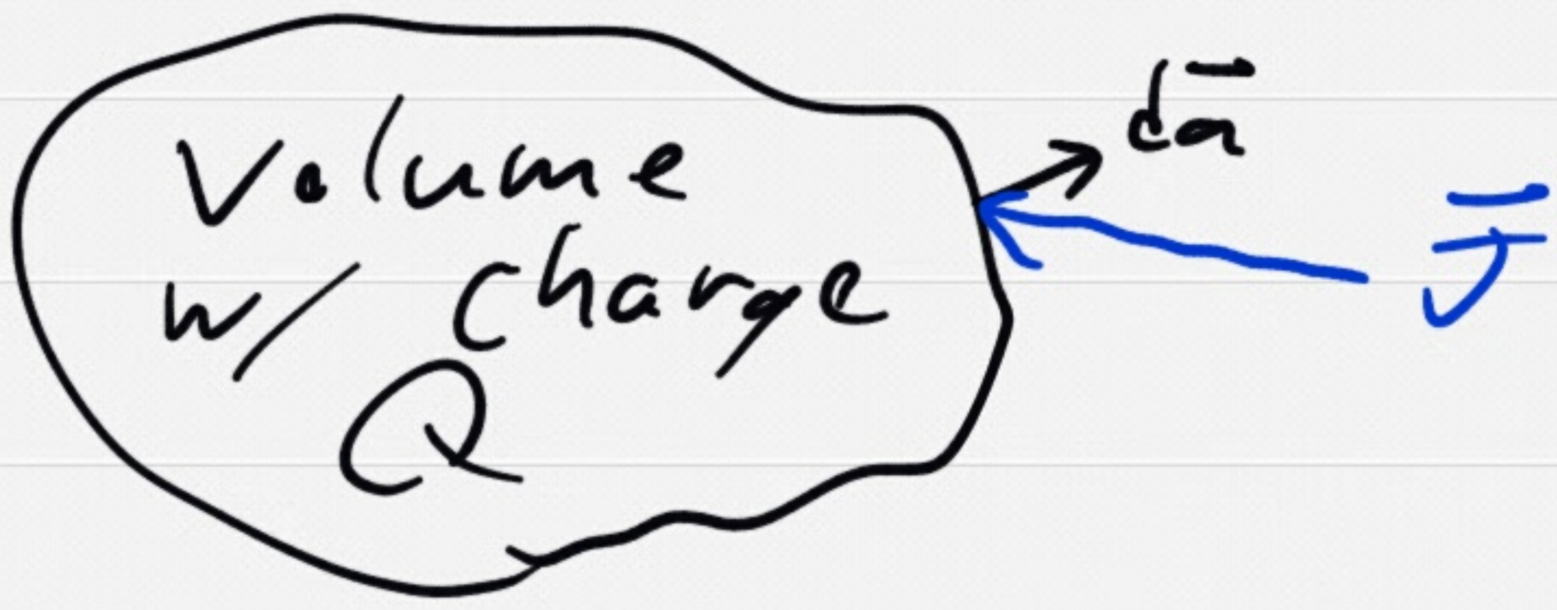
Professor Jasper Halekas
Van Allen 70
MWF 9:30-10:20 Lecture

Ch. 8	Conservation Laws
8.1	Charge & Energy

Continuity of Charge

$$Q(t) = \int_V \rho(\vec{r}, t)\, d\tau$$

$$\vec{J} = d\vec{I}/da_\perp \iff I = \int \vec{J} \cdot d\vec{a}$$



$$dQ/dt = I_{total} = -\oint \vec{J} \cdot d\vec{a}$$

(negative since surface normal inward)

$$\partial/\partial t \int_V \rho(\vec{r}, t)\, d\tau = -\oint \vec{J} \cdot d\vec{a}$$

$$\int_V \partial\rho/\partial t\, d\tau = -\int_V \nabla \cdot \vec{J}\, d\tau$$

$$\Rightarrow \boxed{\partial\rho/\partial t = -\nabla \cdot \vec{J}}$$

Any conserved quantity satisfies a similar equation!

Electromagnetic Energy

$$W_e = \frac{1}{2}\varepsilon_0 \int E^2 \, d\tau$$
$$W_b = \frac{1}{2\mu_0} \int B^2 \, d\tau$$

Guess EM energy density

$$u = \frac{1}{2}\left(\varepsilon_0 E^2 + B^2/\mu_0\right)$$

Proof:

Work on charge δq in time dt

$$\delta dW = \delta \vec{F} \cdot d\vec{l} = \delta q\,(\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{l}$$
$$= \delta q\,(\vec{E} + \vec{v} \times \vec{B}) \cdot \vec{v}\, dt$$
$$= \delta q\, \vec{E} \cdot \vec{v}\, dt$$
$$= \rho\, \delta\tau\, \vec{E} \cdot \vec{v}\, dt$$
$$= \vec{J} \cdot \vec{E}\, \delta\tau\, dt$$

$$\Rightarrow \delta\, dW/dt = \vec{J} \cdot \vec{E}\, \delta\tau$$

$$\Rightarrow \boxed{dW/dt = \int (\vec{J} \cdot \vec{E})\, d\tau}$$

= Energy transfer to particles from fields

EM Energy Transfer

$$\vec{J}\cdot\vec{E} = \left(\frac{\nabla\times\vec{B}}{\mu_0} - \epsilon_0 \partial\vec{E}/\partial t\right)\cdot\vec{E}$$

But $\vec{E}\cdot(\nabla\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{E}) - \nabla\cdot(\vec{E}\times\vec{B})$

$$= \vec{B}\cdot(-\partial\vec{B}/\partial t) - \nabla\cdot(\vec{E}\times\vec{B})$$

$$\Rightarrow \vec{J}\cdot\vec{E} = -\epsilon_0 \vec{E}\cdot\partial\vec{E}/\partial t - \frac{1}{\mu_0}\vec{B}\cdot\partial\vec{B}/\partial t$$
$$-1/\mu_0\, \nabla\cdot(\vec{E}\times\vec{B})$$

$$\boxed{\vec{J}\cdot\vec{E} = -\frac{1}{2}\partial/\partial t\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \nabla\cdot\left(\frac{\vec{E}\times\vec{B}}{\mu_0}\right)}$$

Write as:

$$\boxed{\vec{J}\cdot\vec{E} = -\partial u/\partial t - \nabla\cdot\vec{S}}$$

w/ $\vec{S} = (\vec{E}\times\vec{B})/\mu_0$

= "Poynting vector"

= EM energy flux

$$dW/dt = -\int \partial u/\partial t\, d\tau - \int \nabla\cdot\vec{S}\, d\tau$$

$$= -\int \partial u/\partial t\, d\tau - \oint \vec{S}\cdot d\vec{a}$$

EM energy density change

EM energy flow

If $dW/dt = 0$

(EM energy conserved)

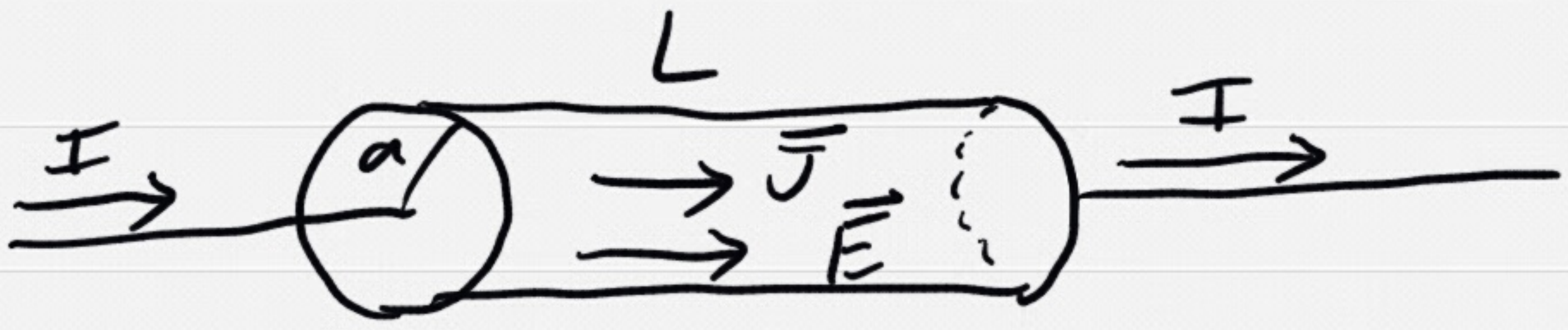
Then $\boxed{\partial u/\partial t = -\nabla \cdot \vec{S}}$

Continuity equation for EM energy

$u \leftrightarrow \rho$

$\vec{S} \leftrightarrow \vec{J}$

Example: Resistor



$$\vec{E} = V/L \,\hat{z}$$

$$\vec{B}(a) = \frac{\mu_0 I}{2\pi a}\,\hat{\varphi}$$

$$\vec{E}\times\vec{B}\Big|_{r=a} = -\frac{\mu_0 I V}{2\pi a L}\,\hat{r}$$

$$\vec{S}(a) = -\frac{IV}{2\pi a L}\,\hat{r}$$

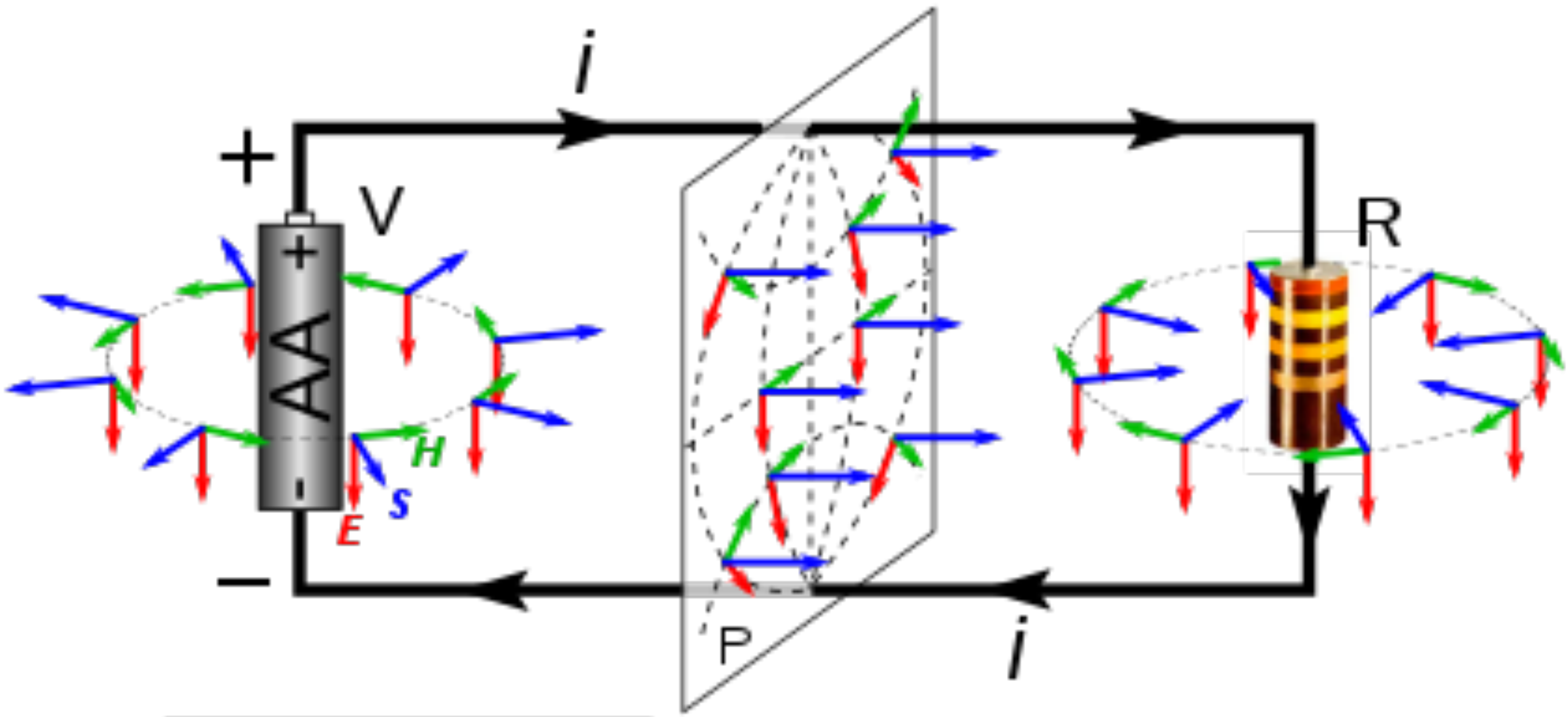
$$dW/dt = -\oint \vec{S}\cdot d\vec{a}$$

$$= -\int_0^{2\pi}\int_0^L -\frac{IV}{2\pi a L}\cdot a\,d\varphi\,dz$$

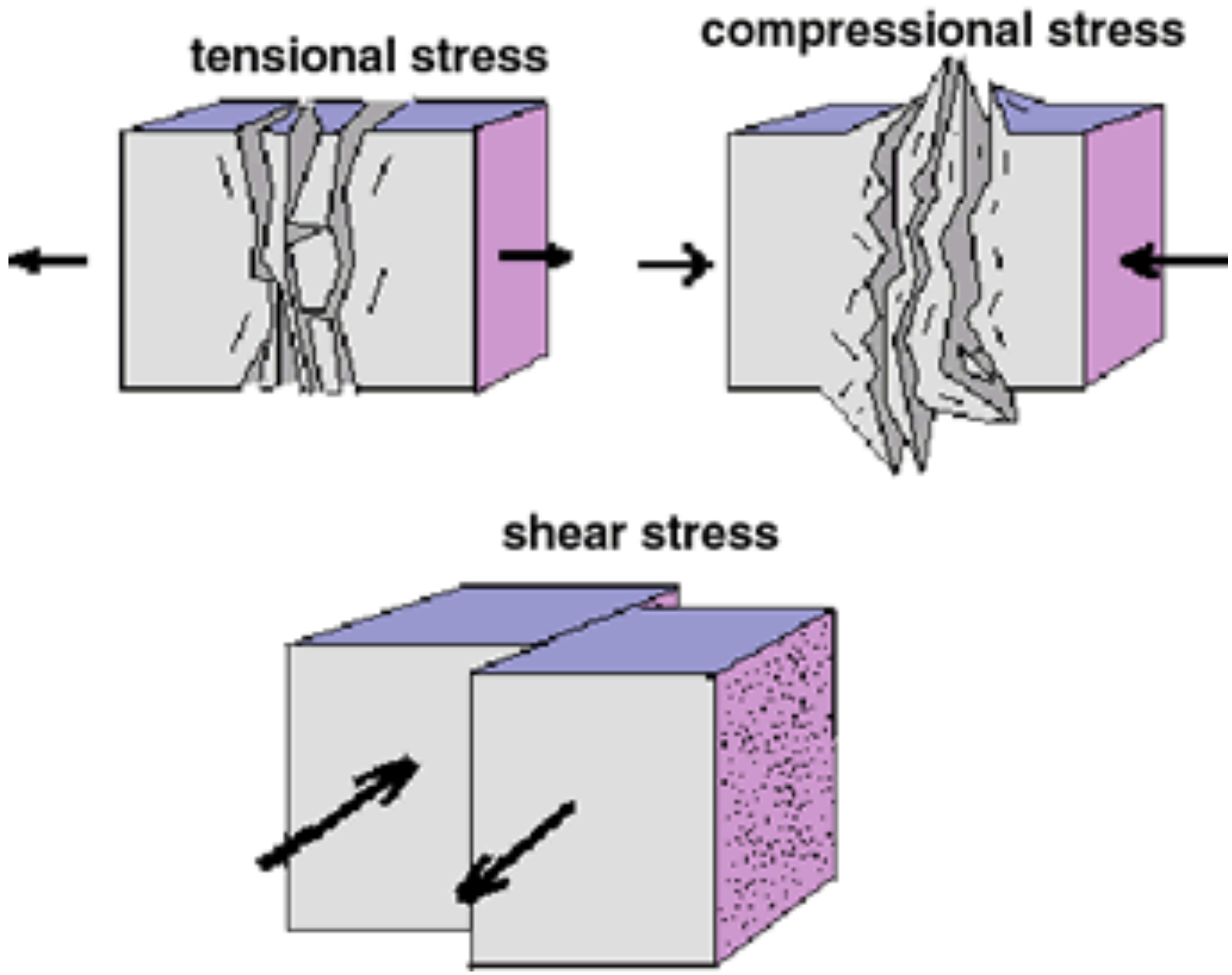
$$= IV$$

- Computed only from fields at surface!!

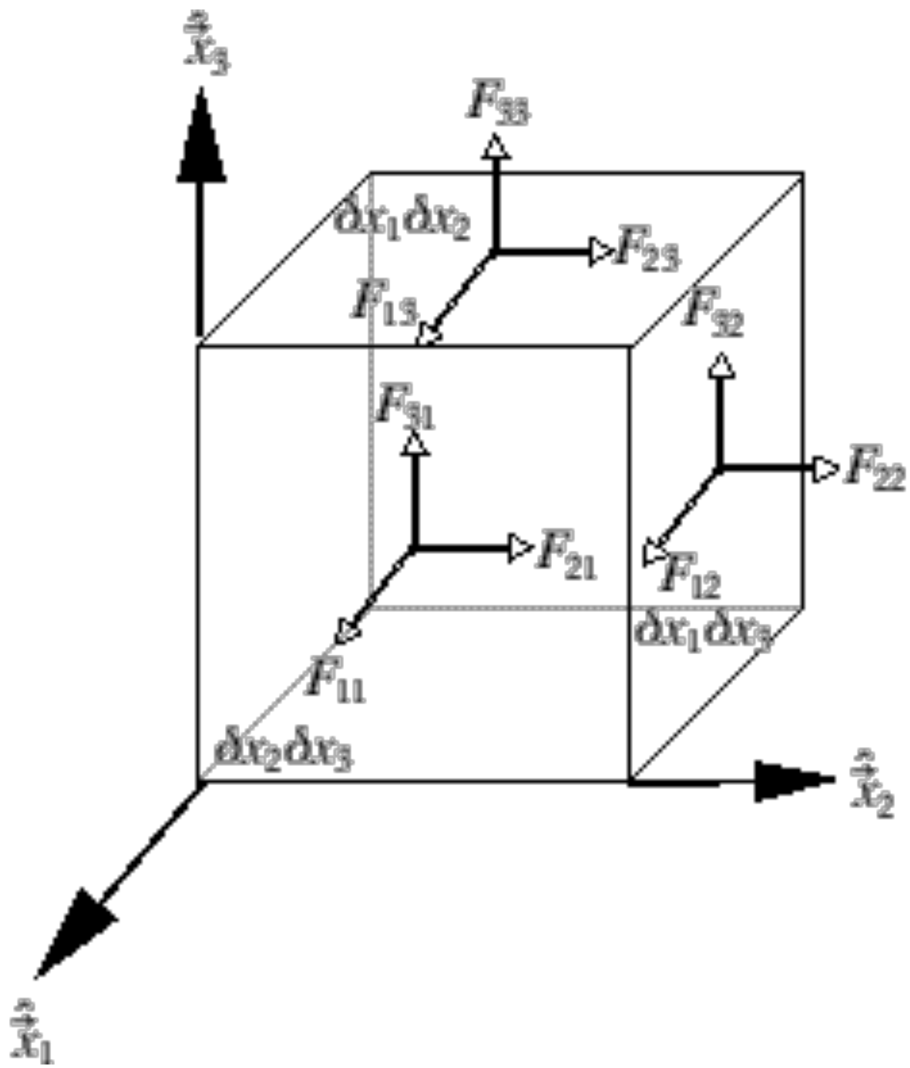
Poynting Flux in Circuit



Pressure & Shear



Stress Tensors



8.2 | Momentum

Force per unit volume

$$\frac{F}{\text{vol}} = \frac{q\vec{E} + q\vec{v}\times\vec{B}}{\text{vol}}$$

$$\Rightarrow \vec{f} = \rho\vec{E} + \vec{J}\times\vec{B}$$

Messy vector math

$$\Rightarrow \vec{f} = \varepsilon_0\left[(\nabla\cdot\vec{E})\vec{E} + (\vec{E}\cdot\nabla)\vec{E}\right] + \frac{1}{\mu_0}\left[(\nabla\cdot\vec{B})\vec{B} + (\vec{B}\cdot\nabla)\vec{B}\right]$$

$$- \nabla\left(\tfrac{1}{2}\varepsilon_0 E^2 + B^2/2\mu_0\right)$$

$$- \mu_0\varepsilon_0\, \partial/\partial t\left((\vec{E}\times\vec{B})/\mu_0\right)$$

"Simplify" using Maxwell Stress Tensor

$$\boxed{T_{ij} = \varepsilon_0\left(E_iE_j - \tfrac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \tfrac{1}{2}\delta_{ij}B^2\right)}$$

$$E_iE_j = \begin{pmatrix} E_x^2 & E_xE_y & E_xE_z \\ E_yE_x & E_y^2 & E_yE_z \\ E_zE_x & E_zE_y & E_z^2 \end{pmatrix}$$

$$\delta_{ij}E^2 = \begin{pmatrix} E^2 & 0 & 0 \\ 0 & E^2 & 0 \\ 0 & 0 & E^2 \end{pmatrix}$$

etc.