- We are used to working with numbers:

$$1, 2, 3, \ldots$$
$$1.1276$$
$$3.14159265 35 \ldots$$
$$x, y, z, \ldots$$

- We can manipulate numbers algebraically:

$$1 + 3 = 4$$
$$x + y = z$$
$$y * z = x$$

- We know how to interpret functions:

$$f(x) = x^2 + 2$$
$$\int f(x)\,dx = \frac{x^3}{3} + 2x + \text{Const.}$$
$$\int_a^b f(x)\,dx = \left(b^3/3 + 2b\right) - \left(a^3/3 + 2a\right)$$
$$df(x)/dx = 2x$$

- All of these are one-dimensional operations. But, the world is (at least) three-dimensional.

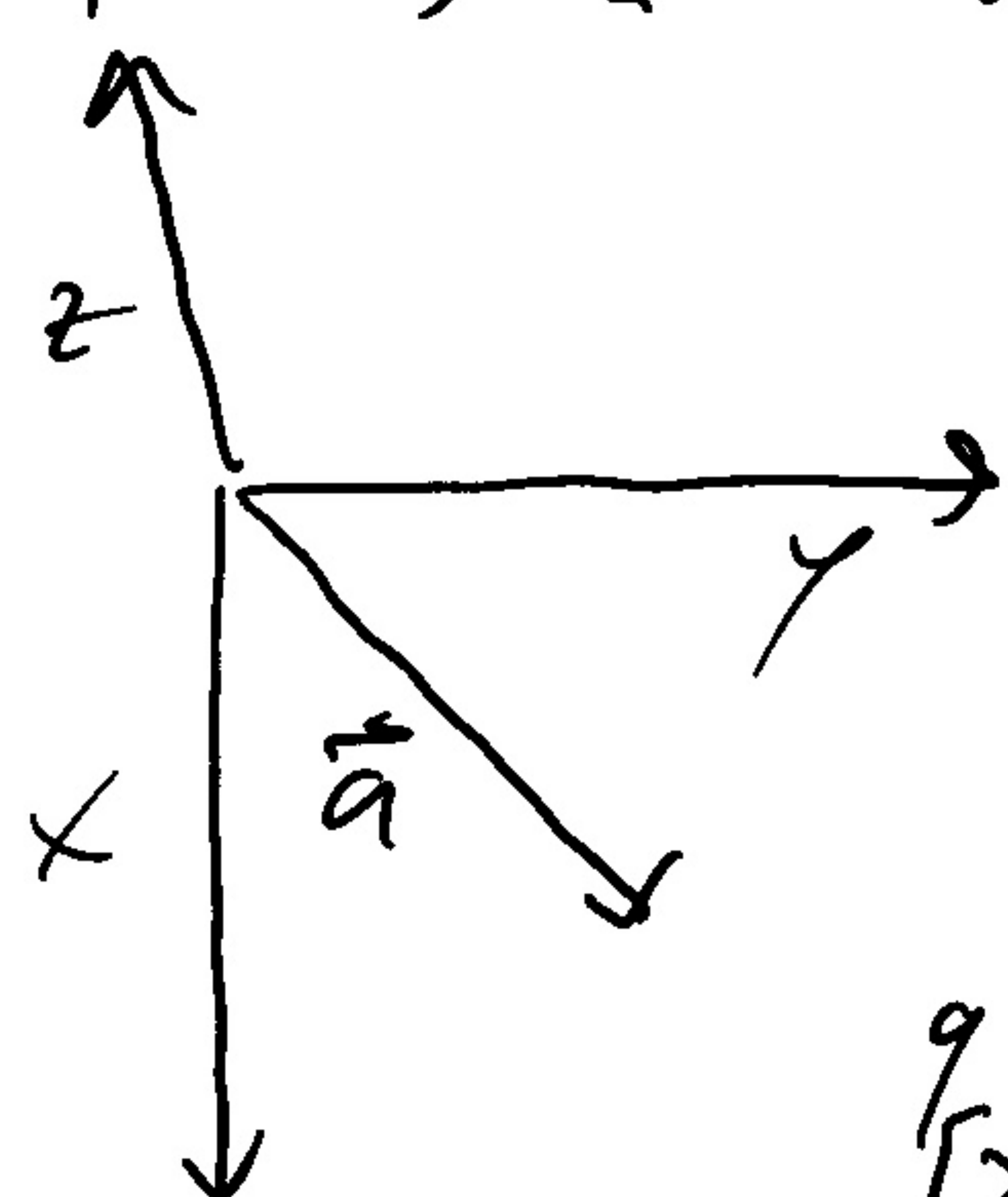
- We need vectors!

- The multi-dimensional equivalent of a number (scalar) is a vector

$\vec{a}, \vec{b}, \vec{c}$ are vectors ($\vec{}$ means a vector)

$\vec{a} = [a_x, a_y, a_z]$ in 3-d

(in other words a vector in 3-d has 3 components)



vector $\vec{a}$ goes from $[x, y, z] = [0, 0, 0]$ to $[a_x, a_y, a_z]$

- origin of coordinates is arbitrary, length and direction are what matter

Length $= |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$

same special vectors

$[1, 0, 0] = \hat{x} = \hat{i}$

$[0, 1, 0] = \hat{y} = \hat{j}$

$[0, 0, 1] = \hat{z} = \hat{k}$

$\hat{}$ means a unit vector, or a vector with length of 1

- can write any vector in terms of unit vectors

$$\vec{a} = a_x \hat{x} + a_y \hat{y} + a_z \hat{z}$$
$$= [a_x, a_y, a_z]$$

- vectors add component by component

$$\vec{a} + \vec{b} = [a_x + b_x, a_y + b_y, a_z + b_z]$$
$$= (a_x + b_x)\hat{x} + (a_y + b_y)\hat{y} + (a_z + b_z)\hat{z}$$

For example:

$$\vec{a} + \vec{b} =$$

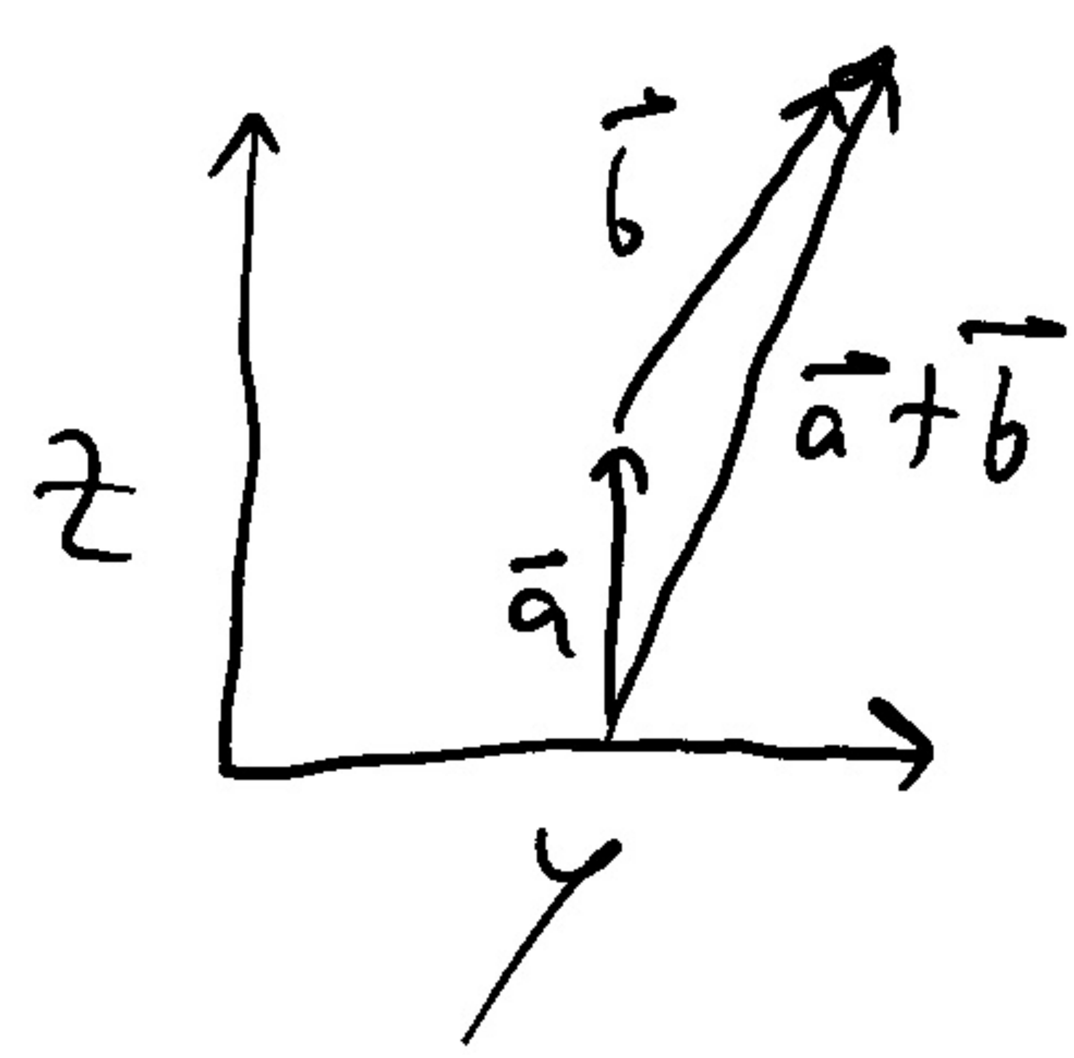
$$[0, 0, 1] + [0, 1, 2]$$

$$= [0, 1, 3]$$

or $\hat{k} + (\hat{j} + 2\hat{k})$

$$= \hat{j} + 3\hat{k}$$

- This has a graphical interpretation:

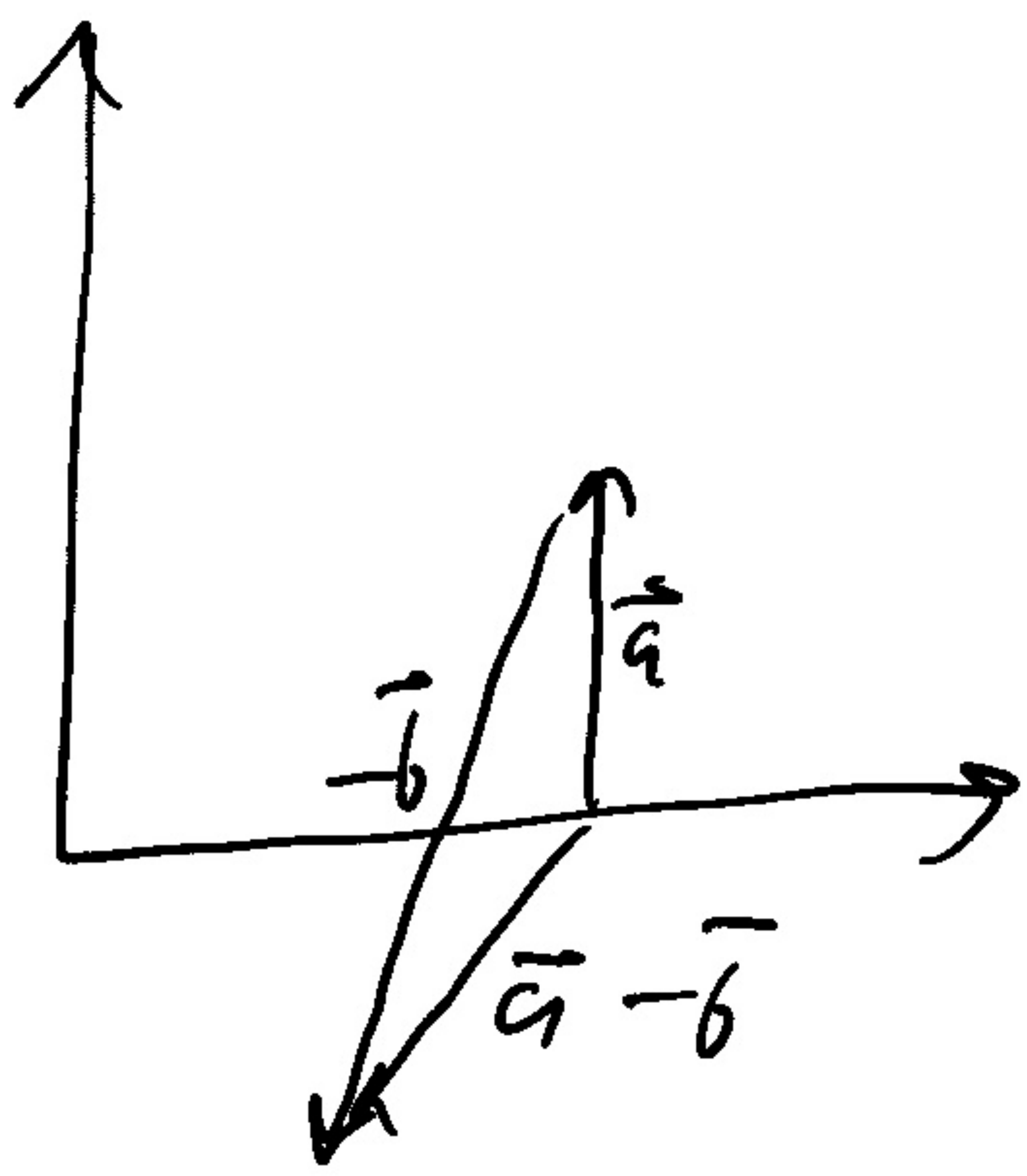


- add head to tail

- Subtraction works too:

$$[0, 0, 1] - [0, 1, 2]$$

$$= [0, -1, -1]$$



flip negative and add head to tail

- Note that the origin is arbitrary for each vector.
- All that matters is direction and magnitude.

- Note $|\vec{a}+\vec{b}| < |\vec{a}|+|\vec{b}|$

In this case:

$$|\vec{a}| = 1$$

$$|\vec{b}| = \sqrt{1^2+2^2} = \sqrt{5} \sim 2.24$$

$$|\vec{a}+\vec{b}| = \sqrt{1^2+3^2} = \sqrt{10} = 3.16 < 3.24$$

What about multiplication?

- Can multiply scalar and vector

$$a\vec{b} = [ab_x, ab_y, ab_z]$$

just multiplies length $|\vec{b}|$ by a

- Can also multiply vector and vector

$$\vec{a}\cdot\vec{b} \neq [a_ib_i, a_jb_j, a_kb_k]$$

This is nonsense

- Two ways to multiply:

$\boxed{\text{dot or scalar product}}$

$$\vec{a}\cdot\vec{b} = a_ib_i + a_jb_j + a_kb_k$$

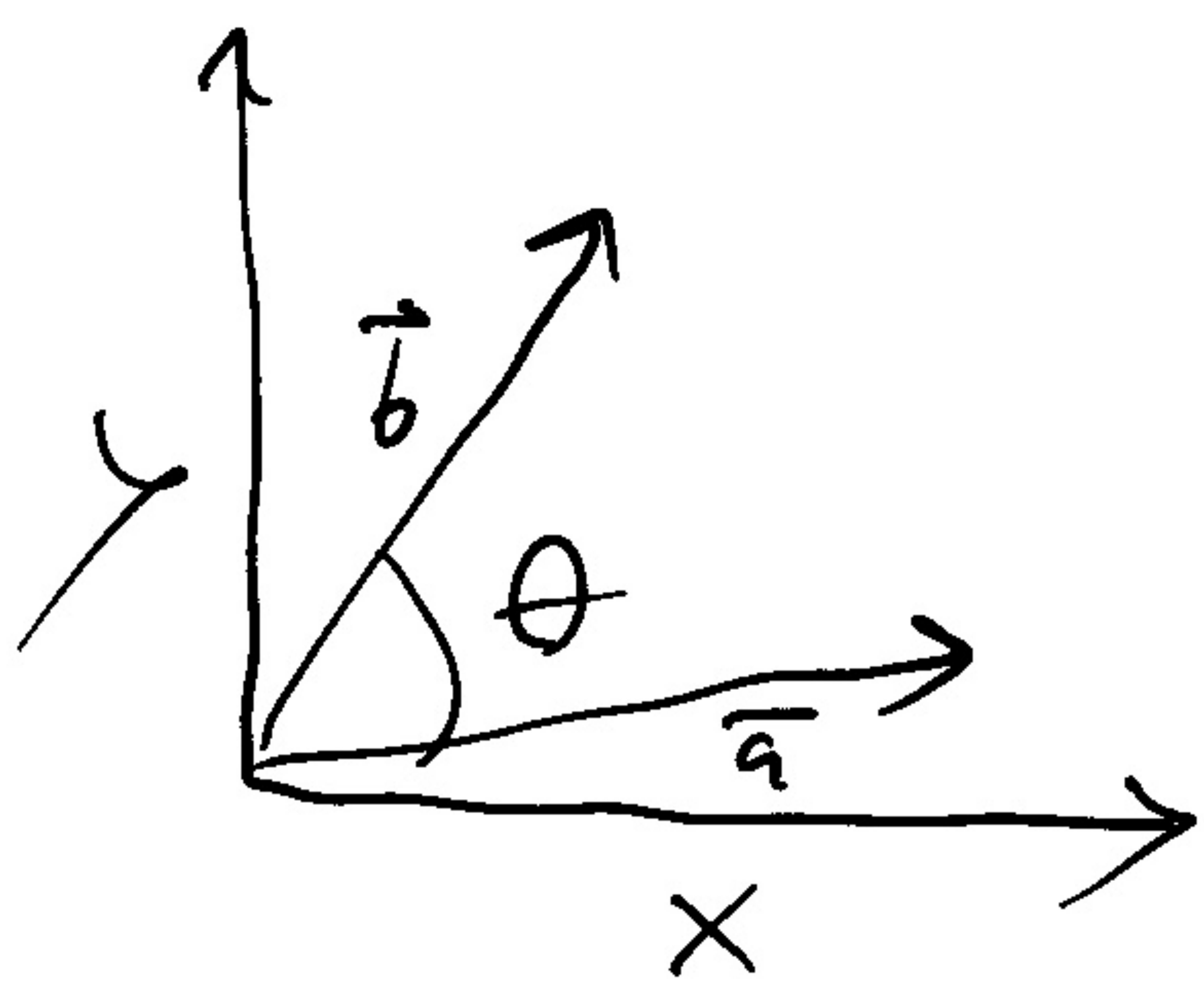
Note: $\vec{a}\cdot\hat{i} = a_i$
$\vec{a}\cdot\hat{j} = a_j$
$\vec{a}\cdot\hat{k} = a_k$

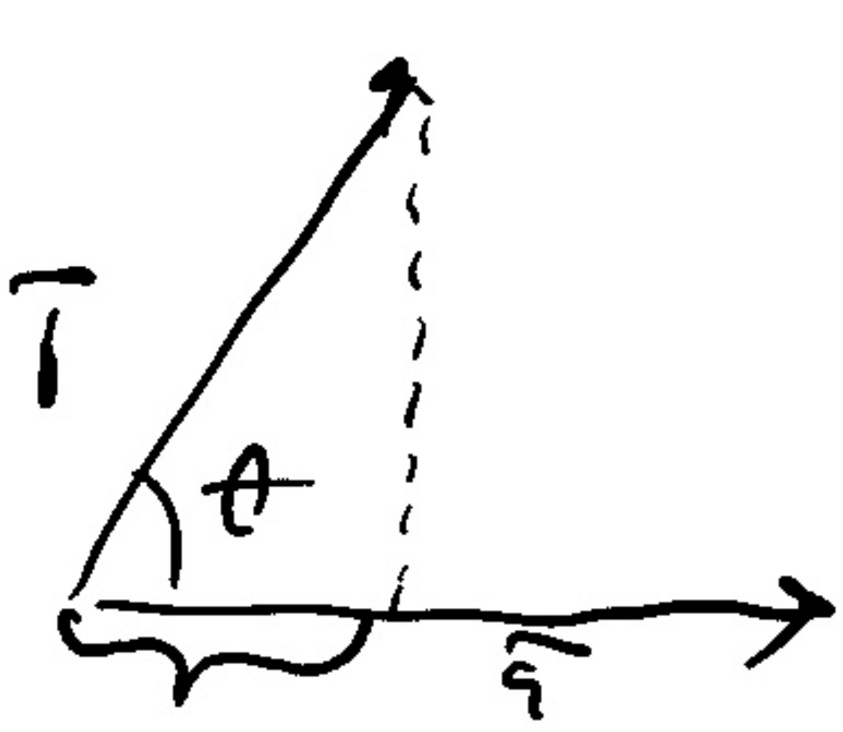
The dot product is a projection.

$\vec{a} \cdot \hat{i}$ is the projection on the x-axis, or the component of $\vec{a}$ parallel to $\hat{x}$

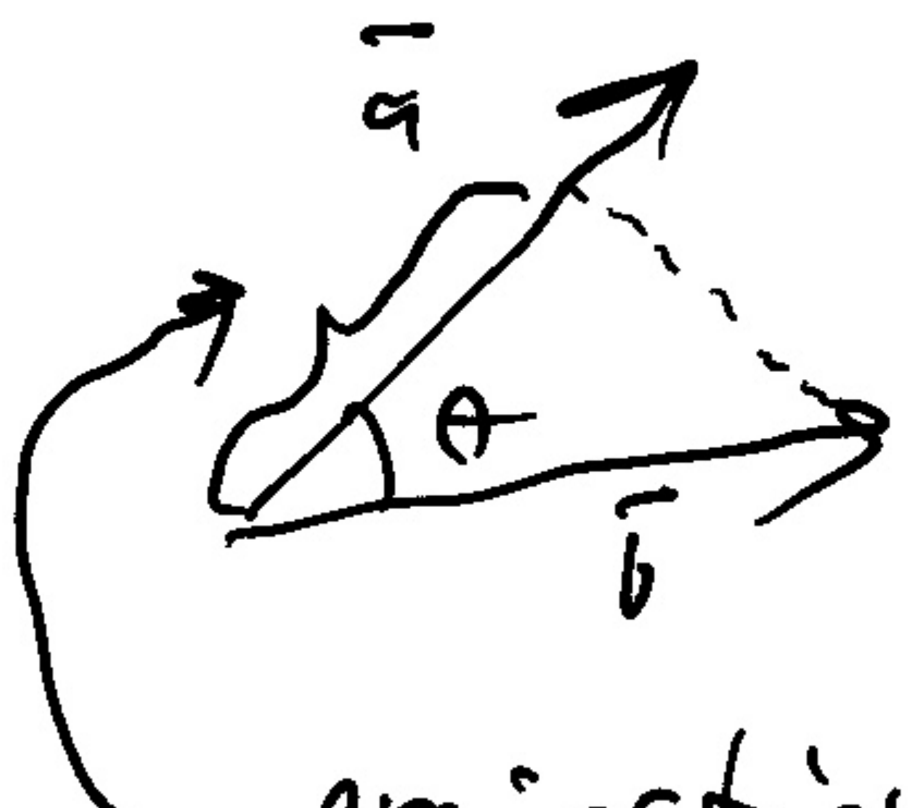
In general:

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}| \cos\theta$$





projection of $\vec{b}$ on $\vec{a}$ $= |\vec{b}| \cos\theta$



projection of $\vec{a}$ on $\vec{b}$ $= |\vec{a}| \cos\theta$

$\vec{a} \cdot \vec{b}$ maximum if

$$\vec{a} \parallel \vec{b}$$

$\vec{a} \cdot \vec{b}$ minimum for

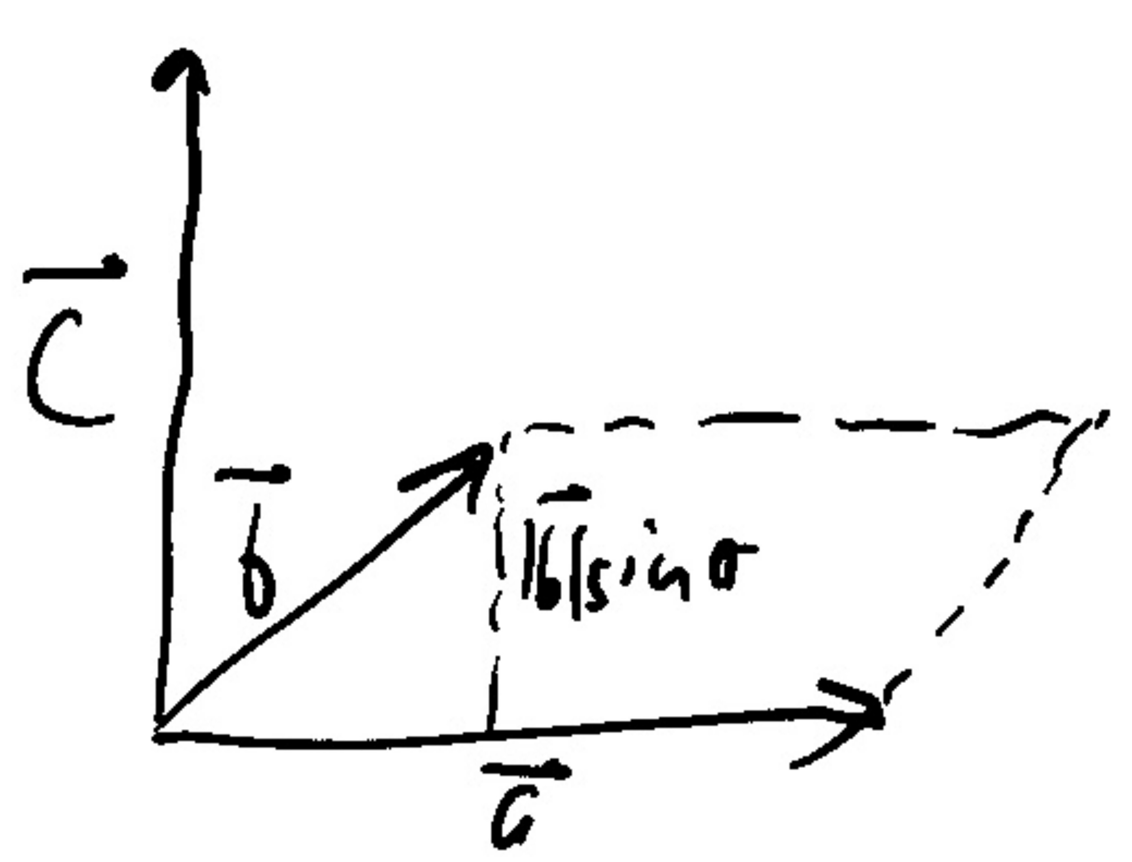
$$\vec{a} \parallel -\vec{b}$$

$\vec{a} \cdot \vec{b} = 0$ for $\vec{a} \perp \vec{b}$

- second way to multiply vectors

$$\boxed{\text{cross or vector product}}$$

First define graphically



$|\vec{a}|\,|\vec{b}| \sin\theta$
= area of parallelogram with sides $\vec{a}, \vec{b}$

$\vec{C} = \vec{a} + \vec{b}$ is perpendicular to $\vec{a}$ and $\vec{b}$ and has magnitude $|\vec{a}|\,|\vec{b}| \sin\theta$

- Direction of $\vec{a} \times \vec{b}$ defined by right hand rule

Fingers along $\vec{a}$ curl towards $\vec{b}$, thumb points in direction of $\vec{a} \times \vec{b}$

$$\hat{i} \times \hat{j} = \hat{k} \qquad \hat{j} \times \hat{i} = -\hat{k}$$
$$\hat{j} \times \hat{k} = \hat{i} \qquad \hat{k} \times \hat{j} = -\hat{i}$$
$$\hat{k} \times \hat{i} = \hat{j} \qquad \hat{i} \times \hat{k} = -\hat{j}$$

- cyclic permutations of ijk are positive, noncyclic are negative

So $\vec{a} \times \vec{b} =$

$$(a_y b_z - a_z b_y)\,\hat{i} + (a_z b_x - a_x b_z)\,\hat{j} + (a_x b_y - a_y b_x)\,\hat{k}$$

use the following trick (if you remember determinants)

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

- Can also have functions of vectors

$$f(\vec{x}) = f(x, y, z)$$

- or vector functions

$$\vec{F}(x) = [F_x(x), F_y(x), F_z(x)]$$

- or even vector functions of vectors

$$\vec{F}(\vec{x}) = [F_x(x,y,z), F_y(x,y,z), F_z(x,y,z)]$$

- The latter are ubiquitous in Electricity & Magnetism